MODULIZATION AND THE ENVELOPING RINGOID

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ABSTRACT. Let A be an algebra in a variety $\mathbf{V}$. We study the modulization of a pointed A-overalgebra P, show that it is totally in any variety that P is totally in, and apply this theory to the construction of the enveloping ringoid $\mathbb{Z}[A, \mathbf{V}]$.

INTRODUCTION

Let A be an algebra of some type Ω. The concept of an *A-module* was first studied in the context of category theory [4], and then generalized [2] in the form of what has come to be known as *Beck modules*. Our own work on this subject [11, 12] includes the definition of $\mathbf{Ab}[A, \mathbf{V}]$, the *category of A-modules totally in the variety* $\mathbf{V}$. This category is equivalent to the earlier-defined category of Beck modules, with some features which we feel are advantageous.

Our previous work also included the construction of the enveloping ringoid $\mathbb{Z}[A, \mathbf{V}]$, over which the left modules form a category isomorphic to $\mathbf{Ab}[A, \mathbf{V}]$. Table 1 shows, in the first column, an algebra in a familiar variety, and in the second and third columns, a familiar ring equivalent to the enveloping ringoid $\mathbb{Z}[A, \mathbf{V}]$. All of these examples are of varieties of algebras with forgetful functors to the variety of groups, although there are additional examples [11, 12] which are not of this type. When a variety does have a forgetful functor to the variety of groups, the enveloping ringoid $\mathbb{Z}[A, \mathbf{V}]$ is always equivalent to a ring, and the equivalence is coherent as A ranges through $\mathbf{V}$ [12].

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$A \in \mathbf{V}$	${}_e\mathbb{Z}[A, \mathbf{V}]_e$	Usual Name
group G	$\mathbb{Z}[G]$	group ring
abelian group G	$\mathbb{Z}$	the integers
group G in a variety of groups $\mathbf{V}_S$	$\mathbb{Z}_S[G]$	universal $\mathbf{V}_S$-envelope [6]
Lie algebra L	$U(L)$	universal enveloping algebra
Jordan algebra J	$U(J)^{\mathrm{op}}$	(universal multiplication envelope)$^{\mathrm{op}}$
commutative ring R	R	R itself
ring R	$R \otimes_{\mathbb{Z}} R^{\mathrm{op}}$	enveloping ring

TABLE 1. Familiar Rings Equivalent to Enveloping Ringoids

In [12], an approach to the definition of the enveloping ringoid was given which used the concept of *modulization*, in which an object called a *pointed A-overalgebra* is made into an A-module. The discussion in [12] was limited to algebras in congruence-modular varieties $\mathbf{V}$. The same construction was used more generally, but without being explicit about applying the modulization functor, in [11]. In this paper, we study modulization in detail for the general (not necessarily congruence-modular) case, and use modulization to construct the enveloping ringoid. In the process, we give a better-motivated construction of the enveloping ringoid than has been given previously.

The purposes of the paper also include describing a calculus of polynomials, with its connections to the theory of pointed A-overalgebras and A-modules, and proving the simplifying fact that if a pointed overalgebra is totally in a variety $\mathbf{V}$, then so is its modulization. Also, a detailed construction of the enveloping ringoid has been given only in [11], and there is some value in publishing it, along with a proof that the categories $\mathbb{Z}[A,\mathbf{V}]$-$\mathbf{Mod}$ and $\mathbf{Ab}[A,\mathbf{V}]$ are not just equivalent, but isomorphic. This has been a point of some confusion.

The plan of the paper is as follows. First there is a section of preliminaries, including discussions of clones, and of ringoids and their modules. Then, in §1 to §4, we define A-modules and pointed A-overalgebras, and introduce related concepts. Next, in §5, we discuss the clone $\mathrm{Pol}(A,\mathbf{V})$ of *polynomials with coefficients in* A, and its relationship to modules and pointed overalgebras. Following that, in §6, we define the ringoid Z_M corresponding to an A-module M, which will later be shown to be a canonical image of the enveloping ringoid of A. In §7, we study the modulization functor, and show that the modulization of a pointed A-overalgebra totally in $\mathbf{V}$ is also totally in $\mathbf{V}$. In §8 and §9, we study the enveloping ringoid, and show that the categories $\mathbb{Z}[A,\mathbf{V}]$-$\mathbf{Mod}$ and $\mathbf{Ab}[A,\mathbf{V}]$ are isomorphic. In §10, we define the canonical ringoid homomorphism from the enveloping ringoid to Z_M. In §11 to §12, we prove that this way of constructing the enveloping ringoid is actually identical to one of the ways given in [11], although the discussion there is not as well motivated. Only in §13 do we consider the hypothesis of congruence-modularity of the variety $\mathbf{V}$, in sketching the simplifications in the theory of the enveloping ringoid which become manifest in that case.

Preliminaries

Category theory. We follow [8] in terminology and notation.

Lattice theory. We will denote the least and greatest elements of a lattice by $\bot$ and $\top$, and if L is a lattice, and a, $b \in L$, the interval sublattice from a to b by $\mathrm{I}_L[a,b]$.

Universal algebra. The basic definitions of Universal Algebra can be found, for example, in [3]. However, we have some preferences in notation and in the way the subject is developed, as follows:

In the definition of an algebra, we prefer to allow an algebra to be empty.

If A is an algebra, we denote its underlying set by $|A|$.

Recall that a *type* is an $\mathbb{N}$-tuple of sets, where $\mathbb{N}$ stands for the set of natural numbers. If Ω is a type, then we call the elements of Ω_n the *n-ary elements* of Ω.

If $\mathbf{V}$ is a variety of algebras of some type, we will typically make no distinction between the category of algebras in $\mathbf{V}$, and $\mathbf{V}$ itself.

$\mathcal{F}_\Omega S$ will stand for the *free algebra* of type Ω on the set S (i.e., the *word algebra*), while $\mathcal{F}_{\mathbf{V}} S$ will stand for its quotient, the *free algebra relative to the variety* $\mathbf{V}$.

We define an n-ary term (of type Ω) to be an element of the algebra $T_n = \mathcal{F}_\Omega\{\, x_1, \dots , x_n \,\}$. If A is an algebra of type Ω, and t is an n-ary term of type Ω, then t^A will stand for the n-ary function on A that sends each n-tuple $\mathbf{a}$ to the image of t, under the unique homomorphism from T_n to A sending each x_i to a_i.

Clones. *Clones* are a part of Universal Algebra which will be important to our development. A *clone* C consists of an $\mathbb{N}$-tuple of sets C_n, called the set of *n-ary elements* of C, together with some additional structure, as follows. For each n, there are elements $\pi^C_{i,n} \in C_n$ as i ranges from 1 to n, called the *i^{th} of n projection*. For each n and n', each n'-tuple of n-ary elements $\mathbf{c}$, and each n'-ary element c', there is an n-ary element $c'\mathbf{c}$, called the *composite* of c' and $\mathbf{c}$. The constants $\pi^C_{i,n}$ and clone composition satisfy

(C1) For all n and all $c \in C_n$, $c\langle \pi^C_{1,n}, \dots , \pi^C_{n,n}\rangle = c$;
(C2) For all n and n', and all i such that $1 \le i \le n$, and for each n-tuple $\mathbf{c}$ of elements of $C_{n'}$, $\pi^C_{i,n}\mathbf{c} = c_i$; and
(C3) For all n, n', and n'', if $\mathbf{c} \in (C_n)^{n'}$, $\mathbf{c}' \in (C_{n'})^{n''}$, and $c'' \in C_{n''}$, we have $c''(\mathbf{c}'\mathbf{c}) = (c''\mathbf{c}')\mathbf{c}$, where $\mathbf{c}'\mathbf{c}$ stands for $\langle c'_1\mathbf{c}, \dots , c'_{n''}\mathbf{c}\rangle$.

If C and C' are clones, then an $\mathbb{N}$-tuple of mappings $\Phi_n : C_n \to C'_n$ is called a *clone homomorphism* if for all n and i, we have $\Phi_n(\pi^C_{i,n}) = \pi^{C'}_{i,n}$ and for all n, n', $\mathbf{c} \in C^{n'}_n$, and $c' \in C_{n'}$, we have $\Phi_n(c'\mathbf{c}) = (\Phi_{n'}(c'))(\Phi_n(\mathbf{c}))$.

There are many natural examples of clones and clone homomorphisms in Universal Algebra. Given a set S, $\operatorname{Clo} S$ will denote the *clone of operations on* S. $\operatorname{Clo}_n S$ is the set of n-ary operations on S, $\pi^{\operatorname{Clo} S}_{i,n}(\mathbf{s}) = s_i$, and given n, n', an n'-tuple of n-ary operations $\mathbf{w}$, an n'-ary operation w', and an n-tuple $\mathbf{s}$ of elements of S, we define $(w'\mathbf{w})(\mathbf{s}) = w'(w_1(\mathbf{s}), \dots , w_{n'}(\mathbf{s}))$. It is straightforward to verify that these definitions make $\operatorname{Clo} S$ a clone.

If Ω is a type, we define $\operatorname{Clo}\Omega$, the *clone of terms of type* Ω, as follows: $\operatorname{Clo}_n \Omega = |T_n|$, the set of n-ary terms of type Ω, defined above. We define $\pi^{\operatorname{Clo}\Omega}_{i,n} = x_i$ and, given n, n', an n'-tuple $\mathbf{t}$ of elements of $\operatorname{Clo}_n \Omega$, and $t' \in \operatorname{Clo}_{n'} \Omega$, we define $t'\mathbf{t}$ to be $(t')^{T_n}(\mathbf{t})$. Note that $\operatorname{Clo}\Omega$ is a free clone on the type Ω; that is, there is a type homomorphism $\Phi_\Omega : \Omega \to \operatorname{Clo}\Omega$ ($\mathbb{N}$-tuple of functions $(\Phi_\Omega)_n : \Omega_n \to \operatorname{Clo}_n \Omega$) such that $\langle \operatorname{Clo}\Omega, \Phi_\Omega\rangle$ is a universal arrow from the type Ω to the forgetful functor from clones to types. $(\Phi_\Omega)_n$ sends $\omega \in \Omega_n$ to $\omega(x_1, \dots , x_n)$, and we have $\omega^A = (\omega(\mathbf{x}))^A$ for every algebra A.

If A is an algebra of type Ω, the mappings $\Phi^A_n : \operatorname{Clo}_n \Omega \to \operatorname{Clo}_n |A|$ given by $t \mapsto t^A$ form a clone homomorphism. In fact, specifying an algebra A is the same as specifying $S = |A|$, the underlying set, and a clone homomorphism from $\operatorname{Clo}\Omega$ to $\operatorname{Clo} S$.

It is important to distinguish between terms and *term operations*, which are operations on the underlying set of an algebra A of the form t^A for some term t. (The term operations on A form a *subclone* of $\operatorname{Clo} |A|$.)

If $\mathbf{V}$ is a variety of algebras of type Ω, then we may contrast $\operatorname{Clo}\Omega$ with $\operatorname{Clo}\mathbf{V}$, the *clone of the variety* $\mathbf{V}$. Elements of $\operatorname{Clo}_n\mathbf{V}$ can be defined as equivalence classes of elements of $\operatorname{Clo}_n\Omega$, under the equivalence relation that relates terms t, t' such that $t^A = (t')^A$ for all $A \in \mathbf{V}$. Or, we can define $\operatorname{Clo}_n\mathbf{V} = \mathcal{F}_{\mathbf{V}}\{\, x_1, \dots , x_n \,\}$. In any case, there is an associated clone homomorphism $\Phi_{\mathbf{V}} : \operatorname{Clo}\Omega \to \operatorname{Clo}\mathbf{V}$. If A is an algebra of type Ω, then we have $A \in \mathbf{V}$ iff Φ^A factors through $\Phi_{\mathbf{V}}$.

For readability, when writing down unary terms, we will write x rather than x_1.

Note for category theorists. Clones are closely related to the category-theoretic concept of *theories* [7]. Given a clone C, we define a category $\bar{C}$ with objects the natural numbers, and such that an arrow from m to n is an n-tuple of elements of C_m. We define 1_n to be $\langle \pi^C_{1,n}, \dots , \pi^C_{n,n}\rangle$. If $\mathbf{c} \in \bar{C}(m, n)$ and $\mathbf{c}' \in \bar{C}(n, p)$, then we define the composition of $\mathbf{c}$ and $\mathbf{c}'$ to be $\mathbf{c}'\mathbf{c}$. Note that conditions (C1), (C2), and (C3) translate very simply to the axioms for a category. $\bar{C}$ is not only a category, but a theory, because each object n is the n-fold direct power of 1. If we construct the theory $\mathbf{T_V}$ corresponding to $\operatorname{Clo}\mathbf{V}$ for some variety $\mathbf{V}$, then an algebra in $\mathbf{V}$ can be thought of as a product-preserving functor from $\mathbf{T_V}$ to the category of sets.

Commutator theory. In §13, which discusses special results applying to the congruence-modular case, we will assume some acquaintance with commutator theory for congruence-modular varieties, as described in [5]. In particular, we will use the notion of a difference term.

Ringoids, and their modules and bimodules. A *ringoid* is a small additive category. That is, a ringoid is a small category X such that the hom-set $X(a, b)$ between two objects a and b is an abelian group, and composition is bilinear. A *ringoid homomorphism* is simply an additive functor, i.e., a functor which is an abelian group homomorphism on each hom-set.

If X is a ringoid, a *left X-module* is an additive functor from X to $\mathbf{Ab}$, the category of abelian groups. Similarly, a *right X-module* is an additive functor from X^{op} to $\mathbf{Ab}$. Homomorphisms of left or right modules are simply natural transformations. We write X-$\mathbf{Mod}$ for the category of left X-modules, and $\mathbf{Mod}$-X for the category of right X-modules.

If X is a ringoid, with set of objects A, we write ${}_{a'}X_a$ rather than $X(a, a')$. We write 1^X_a, or simply 1_a, for the identity element of ${}_aX_a$, and ${}_{a'}0^X_a$, or simply ${}_{a'}0_a$, for the zero element of ${}_{a'}X_a$. We call the composition of arrows *multiplication.* For each $a \in A$, X_a will stand for the left X-module consisting of the abelian groups ${}_bX_a$, and ${}_aX$ will stand for the right X-module consisting of the groups ${}_aX_b$. If Y is another ringoid and $f : X \to Y$ is a ringoid homomorphism, we write ${}_{a'}f_a$ for the abelian group homomorphism from ${}_{a'}X_a$ to ${}_{fa'}Y_{fa}$ defined by the additive functor f. If M is a left X-module, we write ${}_aM$ rather than Ma, and for each $m \in {}_aM$ and $x \in {}_{a'}X_a$, we write xm rather than $(Mx)(m)$. Finally, if M and M' are left X-modules, and $\phi : M \overset{\bullet}{\to} M'$ is a homomorphism, then we simply write $\phi : M \to M'$, and we write ${}_a\phi$ for the component of ϕ at a, rather than ϕa.

Our notation allows us to express the category-theoretic definitions of ringoid theory in a form more reminiscent of ring theory. For example, suppose that X is a ringoid, with set

of objects A, and that M is a left X-module. This means that M is an A-tuple of abelian groups, that for each a and $m \in {}_aM$, $1_a^X m = m$, that xm is $\mathbb{Z}$-bilinear in x and in m, and that for each a, a', $a'' \in A$, each $x \in {}_aX_{a'}$, and each $x' \in {}_{a'}X_{a''}$, $(x'x)m = x'(xm)$.

As an example of a ringoid, let $\{\, {}_aM \,\}_{a\in A}$ be an A-tuple of abelian groups for some index set A. We define the ringoid $\mathrm{End}(M)$ to have set of objects A, and for each a, $a' \in A$, to have hom-set ${}_{a'}\,\mathrm{End}(M)_a = \mathbf{Ab}({}_aM, {}_{a'}M)$. Note that M is a left $\mathrm{End}(M)$-module in an obvious way.

If X is a ringoid, with set of objects A, and J is an A^2-tuple of subgroups ${}_{a'}J_a$ of the respective ${}_{a'}X_a$, then we say that J is an *ideal* of X if for each a, a', and $a'' \in A$, and each $x \in {}_{a'}X_a$ and $x' \in {}_{a''}X_{a'}$, if either $x' \in {}_{a''}J_{a'}$ or $x \in {}_{a'}J_a$, then $x'x \in {}_{a''}J_a$.

If X and X' are ringoids, X has set of objects A, and $f : X \to X'$ is a homomorphism, then we define $\mathrm{Ke}\, f$, the *kernel* of f, to be the A^2-tuple of subgroups of the ${}_{a'}X_a$ defined by ${}_{a'}(\mathrm{Ke}\, f)_a = \mathrm{Ke}\, {}_{a'}f_a$. It is easy to see that $\mathrm{Ke}\, f$ is an ideal of X.

If X is a ringoid, having A as set of objects, and J is an ideal of X, then we can define a new ringoid X/J, again having A as its set of objects, by the formula ${}_{a'}(X/J)_a = {}_{a'}X_a/{}_{a'}J_a$, with identity elements $1_a^{X/J} = 1_a^X/{}_aJ_a$, and with the multiplication well-defined by $(x'/{}_{a''}J_{a'})(x/{}_{a'}J_a) = x'x/{}_{a''}J_a$. We call X/J the *quotient* of X by the ideal J. Clearly, the A^2-tuple of mappings ${}_{a'}X_a \to {}_{a'}(X/J)_a$ defined by $x \mapsto x/{}_{a'}J_a$, form a homomorphism with kernel J, and we denote this homomorphism by $\mathrm{nat}\, J$.

To simplify notation, in what follows, we will typically suppress the subscripts for J. Thus, we will write x/J rather than $x/{}_{a'}J_a$, and ${}_{a'}X_a/J$ rather than ${}_{a'}X_a/{}_{a'}J_a$.

If X is a ringoid, with set of objects A, then we call an A^2-tuple of subgroups ${}_bX'{}_a \subseteq {}_bX_a$ a *subringoid* if $x \in {}_bX_a$, $x' \in {}_cX_b$ imply $x'x \in {}_cX_a$.

Both the ideals and the subringoids of X are partially ordered in an obvious manner, and form complete lattices.

1. Modules and Pointed Overalgebras

Definitions. Let A be an algebra of some type Ω. We define an *abelian group A-overalgebra*, or *A-module*, to be an $|A|$-tuple of abelian groups $\{\, {}_aM \,\}_{a\in A}$, provided with, for each n-ary operation symbol ω and each n-tuple $\mathbf{a}$ of elements of A, a group homomorphism $\omega_{\mathbf{a}}^M : {}_{\mathbf{a}}M \to {}_{\omega(\mathbf{a})}M$, where ${}_{\mathbf{a}}M$ stands for ${}_{a_1}M \times \ldots \times {}_{a_n}M$.

A *pointed A-overalgebra* is the same as an A-module, but with pointed sets ${}_aM$ instead of abelian groups, and pointed set maps $\omega_{\mathbf{a}}^M : {}_{\mathbf{a}}M \to {}_{\omega(\mathbf{a})}M$ rather than abelian group homomorphisms. We write ${}_a{*}^M$, or simply ${}_a{*}$, for the basepoint of ${}_aM$. (The basepoint of ${}_{\mathbf{a}}M$ is $\langle {}_{a_1}{*}, \ldots, {}_{a_n}{*} \rangle$.) An *$A$-overalgebra* is again the same thing, but with sets ${}_aM$ rather than pointed sets or abelian groups. Finally, an *A-set* is simply an $|A|$-tuple of sets.

Homomorphisms. If S and T are A-sets, we call an $|A|$-tuple of functions ${}_af : {}_aS \to {}_aT$ an *A-function* and write $f : S \to T$. If M and M' are A-modules, a *homomorphism* $\phi : M \to M'$ is an A-function, the components of which are abelian group homomorphisms, such that for

each n-ary operation symbol ω, each n-tuple $\mathbf{a}$ of elements of A, and each $\mathbf{m} \in {}_{\mathbf{a}}M$, we have

$$_{\omega(\mathbf{a})}\phi(\omega^M_{\mathbf{a}}(\mathbf{m})) = \omega^{M'}_{\mathbf{a}}({}_{a_1}\phi(m_1), \dots , {}_{a_n}\phi(m_n)).$$

A *homomorphism of pointed A-overalgebras* is the same thing, except that it is an A-tuple of pointed set maps. A *homomorphism of A-overalgebras* is again the same, but the functions do not have abelian group structures or basepoints to preserve.

We denote the category of A-modules by $\mathbf{Ab}[A]$, the category of pointed A-overalgebras by $\mathbf{Pnt}[A]$, and the category of A-overalgebras by $\mathbf{Ov}[A]$.

$\mathbf{Ab}[A]$ is an abelian category, if given ϕ, $\phi' \in \mathbf{Ab}[A](M, M')$, we define ${}_a(\phi+\phi') = {}_a\phi+{}_a\phi'$, ${}_a(-\phi) = -{}_a\phi$, and ${}_a0 = 0 \in \mathbf{Ab}({}_aM, {}_aM')$. See §4 for descriptions of the kernel and cokernel of a homomorphism.

Example 1.1. Let B be an algebra of the same type as A, and let $\pi : B \to A$ and $\iota : A \to B$ be homomorphisms such that $\pi\iota = 1_A$. Then we define the pointed A-overalgebra $[\![B, \pi, \iota]\!]$ by ${}_a[\![B, \pi, \iota]\!] = \pi^{-1}(a)$, ${}_a{*} = \iota(a)$, and $\omega^{[\![B,\pi,\iota]\!]}_{\mathbf{a}}(\mathbf{b}) = \omega^B(\mathbf{b})$.

Example 1.2. Let $\beta \in \operatorname{Con} A$. We define $\beta^* = [\![A(\beta), \pi, \iota]\!]$, where $A(\beta)$ is the subalgebra of A^2 given by pairs of elements related by β, $\pi : \langle a, a'\rangle \mapsto a$, and $\iota : a \mapsto \langle a, a\rangle$. More generally, if α, $\beta \in \operatorname{Con} A$ are such that $\alpha \le \beta$, we define $\mathrm{P}[\alpha, \beta]$ by ${}_a\mathrm{P}[\alpha, \beta] = \{\, a'/\alpha \mid a \ \beta \ a' \,\}$, ${}_a{*}^{\mathrm{P}[\alpha,\beta]} = a/\alpha$, and $\omega^{\mathrm{P}[\alpha,\beta]}_{\mathbf{a}}(\mathbf{x}) = \omega^{A/\alpha}(\mathbf{x})$.

Beck modules. $\mathbf{Ab}[A]$ is equivalent to the category of *Beck modules* over A [2]. The reason for this equivalence is that $\mathbf{Ab}[A]$ is the category of abelian group objects of $\mathbf{Ov}[A]$, the category of Beck modules is the category of abelian group objects of (Ω-$\mathbf{Alg} \downarrow A$), the category of algebras (of type Ω) over A, and $\mathbf{Ov}[A]$ is equivalent to (Ω-$\mathbf{Alg} \downarrow A$). (Note that the term *Beck module* also encompasses abelian group objects in the category ($\mathbf{V} \downarrow A$) where $\mathbf{V}$ is a variety of algebras; we will define an equivalent category $\mathbf{Ov}[A, \mathbf{V}]$ based on $\mathbf{Ov}[A]$ in the next section.)

There are two basic reasons for working in categories derived from $\mathbf{Ov}[A]$ rather than in those derived from (Ω-$\mathbf{Alg} \downarrow A$). One reason is that it is very useful to be able to define A-modules and similar objects M such that the ${}_aM$ are not necessarily disjoint. The other reason is that if we were to define, for example, an A-module M, in terms of an algebra B with a homomorphism $f : B \to A$, we would still need to talk about the abelian groups $f^{-1}(a)$ for $a \in A$, which we of course denote by ${}_aM$. And, indeed, the abelian group structure (or, pointed set structure, if we talk about pointed overalgebras) is central to the theory. So it seems that we would need to talk about the ${}_aM$ in any case. Our definition and notation focus attention where we argue it should be.

In any case, results concerning Beck modules certainly apply to the category of A-modules as we define it.

We mention also that pointed overalgebras have been studied before [9, 10, 13], in the form of pointed set objects of ($\mathbf{V} \downarrow A$).

2. The Total Algebra Construction; $\mathbf{Ab}[A, \mathbf{V}]$ and $\mathbf{Pnt}[A, \mathbf{V}]$

Let M be an A-module, pointed A-overalgebra, or A-overalgebra. We define the *total algebra* of A (called by some authors the *semidirect product*) to be the set $A \ltimes M = \{ \langle a, m\rangle \mid m \in {}_aM \}$, provided with operations defined by $\omega^{A\ltimes M}(\langle a_1, m_1\rangle, \ldots, \langle a_n, m_n\rangle) = \langle \omega(\mathbf{a}), \omega^M_{\mathbf{a}}(\mathbf{m})\rangle$ for each n-ary operation symbol ω.

Associated with the total algebra is a homomorphism $\pi_M : A \ltimes M \to A$, defined by $\pi_M : \langle a, m\rangle \mapsto a$. In the case of an A-module or pointed A-overalgebra, there is also a homomorphism $\iota_M : A \to A \ltimes M$, defined by $a \mapsto \langle a, {}_a0\rangle$ (or, for M a pointed A-overalgebra, by $a \mapsto \langle a, {}_a{*}\rangle$.) Note that $\pi_M \iota_M = 1_A$, and that π_M is onto in these cases.

In the case of an A-set S, we can still define $A \ltimes S$ and π_S, but they are only a set and a function, rather than an algebra and a homomorphism.

Modules, pointed overalgebras, and overalgebras totally in a variety $\mathbf{V}$. Let A belong to a variety $\mathbf{V}$. Then we say that an A-module, pointed A-overalgebra, or A-overalgebra M is *totally in* $\mathbf{V}$ if $A \ltimes M$ belongs to $\mathbf{V}$. We denote the full subcategory of $\mathbf{Ab}[A]$ of A-modules (pointed A-overalgebras, A-overalgebras) totally in $\mathbf{V}$ by $\mathbf{Ab}[A, \mathbf{V}]$ (respectively, by $\mathbf{Pnt}[A, \mathbf{V}]$, $\mathbf{Ov}[A, \mathbf{V}]$).

Remark. In example 1.1, $[\![B, \pi, \iota]\!] \in \mathbf{V}$ if $B \in \mathbf{V}$. In example 1.2, if α, $\beta \in \operatorname{Con} A$ for $A \in \mathbf{V}$, then $\beta^* \in \mathbf{V}$ and $\mathrm{P}[\alpha, \beta] \in \mathbf{V}$.

Example 2.1 (Free pointed overalgebras)*.* There is an evident forgetful functor

$$\mathcal{U}_A : \mathbf{Pnt}[A, \mathbf{V}] \to A\text{-}\mathbf{Set}.$$

If S is an A-set, then let $B = A \coprod \mathcal{F}_{\mathbf{V}}(A \ltimes S)$, and $P = [\![B, \pi, \iota]\!]$, where $\iota : A \to B$ is the insertion into the coproduct, and $\pi : B \to A$ is the homomorphism determined by 1_A, $\pi_S : A \ltimes S \to |A|$, and the universal property of the relatively free algebra. Then $\langle P, \xi\rangle$ is a universal arrow from S to $\mathcal{U}_A$, where $\xi : S \to \mathcal{U}_A P$ is defined by ${}_a\xi : s \mapsto \langle a, s\rangle \in {}_aP$. We say that P is *free on S (relative to $\mathbf{V}$).*

3. A-operations

If w is an n-ary operation on a set A, and M is an A-tuple of abelian groups, pointed sets, or simply sets, we call an A^n-tuple W of abelian group or pointed set homomorphisms (or, in the case of an A-tuple of sets, functions) $W_{\mathbf{a}} : {}_{\mathbf{a}}M \to {}_{w(\mathbf{a})}M$ an *A-operation on M, over w*. If we denote the disjoint union of the ${}_aM$ by $A \ltimes M$, then besides possibly consisting of homomorphisms of abelian groups or of pointed sets, W consists of exactly the information needed to specify an n-ary operation, which we denote by $A \ltimes W$, on $A \ltimes M$, such that the obvious function $\pi : A \ltimes M \to A$ is a homomorphism from the algebra $\langle A \ltimes M, A \ltimes W\rangle$ to the algebra $\langle A, w\rangle$.

We define $\operatorname{Clo}_n M$ to the the set of pairs $\langle w, W\rangle$ such that w is an n-ary operation on A, and W is an A-operation on M over w. We set $\pi_{in}^{\operatorname{Clo} M} = \langle \pi_{in}^{\operatorname{Clo} A}, \tilde{\pi}_{in}\rangle$, where $(\tilde{\pi}_{in})_{\mathbf{a}}(\mathbf{m}) = m_i$, and given $\langle w_1, W_1\rangle$, …, $\langle w_{n'}, W_{n'}\rangle \in \operatorname{Clo}_n M$ and $\langle w', W'\rangle \in \operatorname{Clo}_{n'} M$, we define the composite

to be $\langle w'\mathbf{w}, W' \circ_{\mathbf{w}} \mathbf{W}\rangle$, where $W' \circ_{\mathbf{w}} \mathbf{W}$ is the A-operation on M over $w'\mathbf{w}$ such that for each n-tuple $\mathbf{a}$ of elements of A, and n-tuple $\mathbf{m}$ of elements of ${}_{\mathbf{a}}M$,

$$(W' \circ_{\mathbf{w}} \mathbf{W})_{\mathbf{a}}(\mathbf{m}) = W'_{\langle w_1(\mathbf{a}),\dots,w_{n'}(\mathbf{a})\rangle}((W_1)_{\mathbf{a}}(\mathbf{m}), \dots, (W_{n'})_{\mathbf{a}}(\mathbf{m})).$$

Notation. In what folows, we will abbreviate $\langle w_1(\mathbf{a}), \dots, w_{n'}(\mathbf{a})\rangle$ by $\mathbf{w}(\mathbf{a})$.

Theorem 3.1. *We have*

1. $\operatorname{Clo} M$ *is a clone; and*
2. *the* $\mathbb{N}$*-tuple of functions defined by* $\langle w, W\rangle \mapsto w$ *form a clone homomorphism* $\Lambda_M : \operatorname{Clo} M \to \operatorname{Clo} A$.

Proof. The only nonobvious part is to show that if $\langle w', W'\rangle \in \operatorname{Clo}_{n'} M$, and $\langle w_1, W_1\rangle$, ..., $\langle w_{n'}, W_{n'}\rangle \in \operatorname{Clo}_n M$, then $\langle w'\mathbf{w}, W' \circ_{\mathbf{w}} \mathbf{W}\rangle \in \operatorname{Clo}_n M$, which, in case M is an A-tuple of abelian groups or pointed sets, requires showing that for each $\mathbf{a} \in A^n$, $(W' \circ_{\mathbf{w}} \mathbf{W})_{\mathbf{a}}$ is a homomorphism. However, $(W' \circ_{\mathbf{w}} \mathbf{W})_{\mathbf{a}}$ is the composition of two homomorphisms: $W'_{\mathbf{w}(\mathbf{a})}$, and the homomorphism from ${}_{\mathbf{a}}M$ to the product ${}_{\mathbf{w}(\mathbf{a})}M$ determined by the $W_{\mathbf{a}} : {}_{\mathbf{a}}M \to {}_{w_i(\mathbf{a})}M$. □

We call $\operatorname{Clo} M$ the *clone of A-operations on M*.

Preservation of A-operations. Let A be a set, and let M, M' be A-tuples of abelian groups, pointed sets, or simply sets. Let $w \in \operatorname{Clo}_n A$ and let W, $\bar{W}$ be such that $\langle w, W\rangle \in \operatorname{Clo}_n M$ and $\langle w, \bar{W}\rangle \in \operatorname{Clo}_n M'$. If ϕ is an A-tuple of abelian group homomorphisms (pointed set homomorphisms, functions) ${}_a\phi : {}_aM \to {}_aM'$, then we say that ϕ *sends W to $\bar{W}$* if for all $\mathbf{a} \in A^n$ and all $\mathbf{m} \in {}_{\mathbf{a}}M$ we have

$${}_{w(\mathbf{a})}\phi(W_{\mathbf{a}}(\mathbf{m})) = \bar{W}_{\mathbf{a}}({}_{a_1}\phi(m_1), \dots, {}_{a_n}\phi(m_n)).$$

Of course, we have seen a similar equation before in the definition of homomorphisms $\phi : M \to M'$ of A-modules, pointed A-overalgebras, and A-overalgebras, when A is an algebra; in such cases, ϕ is a homomorphism iff ϕ sends ω^M to $\omega^{M'}$ for each ω.

Theorem 3.2. *Let A be a set, M, M' A-tuples of abelian groups, pointed sets, or simply sets, $w' \in \operatorname{Clo}_{n'} A$, w_1, ..., $w_{n'} \in \operatorname{Clo}_n A$, and W', $\bar{W}'$, W_1, ..., $W_{n'}$, $\bar{W}_1$, ..., $\bar{W}_{n'}$ be such that $\langle w', W'\rangle \in \operatorname{Clo}_{n'} M$, $\langle w', \bar{W}'\rangle \in \operatorname{Clo}_{n'} M'$, and $\langle w_i, W_i\rangle \in \operatorname{Clo}_n M$ and $\langle w_i, \bar{W}_i\rangle \in \operatorname{Clo}_n M'$ for all i. Let ϕ be an A-tuple of abelian group homomorphisms, pointed set homomorphisms, or functions, respectively. If ϕ sends W' to $\bar{W}'$ and W_i to $\bar{W}_i$ for all i, then ϕ sends $W' \circ_{\mathbf{w}} \mathbf{W}$ to $\bar{W}' \circ_{\mathbf{w}} \bar{\mathbf{W}}$.*

Proof. For each $\mathbf{a} \in A^n$, and each $\mathbf{m} \in {}_{\mathbf{a}}M$, we have

$$\begin{aligned}
{}_{w'\mathbf{w}(\mathbf{a})}\phi((W' \circ_{\mathbf{w}} \mathbf{W})_{\mathbf{a}}(\mathbf{m})) &= {}_{w'\mathbf{w}(\mathbf{a})}\phi(W'_{\mathbf{w}(\mathbf{a})}((W_1)_{\mathbf{a}}(\mathbf{m}), \dots, (W_{n'})_{\mathbf{a}}(\mathbf{m}))) \\
&= \bar{W}'_{\mathbf{w}(\mathbf{a})}(\dots, {}_{w_i(\mathbf{a})}\phi((W_i)_{\mathbf{a}}(\mathbf{m})), \dots) \\
&= \bar{W}'_{\mathbf{w}(\mathbf{a})}(\dots, (\bar{W}_i)_{\mathbf{a}}({}_{a_1}\phi(m_1), \dots, {}_{a_n}\phi(m_n)), \dots) \\
&= (\bar{W}' \circ_{\mathbf{w}} \bar{\mathbf{W}})_{\mathbf{a}}({}_{a_1}\phi(m_1), \dots, {}_{a_n}\phi(m_n)).
\end{aligned}$$

□

The A-operations t^M. Let A be an algebra, and let M be an A-module, pointed A-overalgebra, or A-overalgebra. As we mentioned before, $\operatorname{Clo}\Omega$ is a free clone on the type Ω. We have an $\mathbb{N}$-tuple of mappings sending n-ary operation symbols ω to $\langle \omega^A, \omega^M\rangle \in \operatorname{Clo} M$. By the universal property of $\operatorname{Clo}\Omega$, there is a corresponding clone homomorphism $\Phi^M : \operatorname{Clo}\Omega \to \operatorname{Clo} M$. Thus, for each n-ary term t, there is an n-ary A-operation t^M over t^A such that $\Phi^M_n(t) = \langle t^A, t^M\rangle$.

Theorem 3.3. *Let A be an algebra of type Ω, and M an A-module, pointed A-overalgebra, or A-overalgebra. We have*

1. $\Lambda_M \Phi^M = \Phi^A$*;*
2. $(\omega(\mathbf{x}))^M = \omega^M$*;*
3. *for each n-ary term t, and n-tuple $\langle a_1, m_1\rangle, \ldots, \langle a_n, m_n\rangle$ of elements of $A \ltimes M$, we have* $t^{A\ltimes M}(\langle a_1, m_1\rangle, \ldots, \langle a_n, m_n\rangle) = \langle t^A(\mathbf{a}), t^M(\mathbf{m})\rangle$*;*
4. *if M' is another A-module, pointed A-overalgebra, or A-overalgebra, $\phi : M \to M'$ is a homomorphism, and $t \in \operatorname{Clo}_n \Omega$, then ϕ sends t^M to $t^{M'}$; and*
5. *if B is an algebra of the same type as A, and $\pi : B \to A$ and $\iota : A \to B$ are homomorphisms such that $\pi\iota = 1_A$, then for each n-tuple of elements of A, and each $\mathbf{b} \in {}_{\mathbf{a}}[\![B, \pi, \iota]\!]$, we have $t^{[\![B,\pi,\iota]\!]}_{\mathbf{a}}(\mathbf{b}) = t^B(\mathbf{b})$.*

A-operations, identities, and varieties.

Theorem 3.4. *Let M be an A-module, pointed A-overalgebra, or A-overalgebra. We have*

1. *If t and t' are n-ary terms such that A satisfies the identity $t = t'$, then $t^M = (t')^M$ iff $t^{A\ltimes M} = (t')^{A\ltimes M}$; and*
2. *M is totally in a variety $\mathbf{V}$ iff $\Phi^M = \Phi'\Phi_{\mathbf{V}}$ for some clone homomorphism $\Phi' : \operatorname{Clo}\mathbf{V} \to \operatorname{Clo} M$, i.e. iff Φ^M factors through $\Phi_{\mathbf{V}} : \operatorname{Clo}\Omega \to \operatorname{Clo}\mathbf{V}$.*

In case the equivalent conditions of part (1) of the theorem are satisfied, we say that M *satisfies the identity* $t = t'$.

4. Subobjects and Quotients

Factorization of homomorphisms. If $\phi : M \to M'$ is a homomorphism of A-modules, we can construct a new A-module $\operatorname{Im}\phi$, the *image* of ϕ, by defining ${}_a\operatorname{Im}\phi = \operatorname{Im}{}_a\phi$ for all a and letting the A-operations be the restrictions of those of M'. ϕ then factors in an obvious way as $\phi_m\phi_e$, where $\phi_e : M \to \operatorname{Im}\phi$ and $\phi_m : \operatorname{Im}\phi \to M'$ are A-module homomorphisms such that for each a, ${}_a\phi_e$ is onto and ${}_a\phi_m$ is one-one. We say that ϕ_e is *onto* and that ϕ_m is *one-one.*

The onto homomorphisms of A-modules form a subcategory $\mathbf{E}$ of $\mathbf{Ab}[A]$, and the one-one homomorphisms form a subcategory $\mathbf{M}$. The pair $\langle \mathbf{E}, \mathbf{M}\rangle$ is an example of a factorization system in $\mathbf{Ab}[A]$. (See [1] for a definition of this concept.)

Similar definitions and remarks apply to homomorphisms of pointed A-overalgebras, A-overalgebras, and A-sets.

Subobjects. If M, M' are A-modules (pointed A-overalgebras, A-overalgebras, A-sets) then we say that $M' \leq M$, or that M' is a *submodule* (respectively, *sub pointed overalgebra*, *sub overalgebra*, *A-subset*) of M, if ${}_aM' \subseteq {}_aM$ for every a. For a given M, the subobjects form a complete lattice, which we denote by $\operatorname{Sub} M$.

On the other hand, given two one-one homomorphisms ι, ι' of A-modules (or pointed overalgebras, etc.) with codomain M, we say that $\iota \leq \iota'$ if there is a homomorphism ϕ such that $\iota = \iota'\phi$. This results in a preorder, and as usual with a preorder, we say that ι and ι' are *equivalent* if $\iota \leq \iota'$ and $\iota' \leq \iota$. The resulting partially-ordered set of equivalence classes is isomorphic to $\operatorname{Sub} M$.

We will speak of A-submodules of a given module M, *generated by* an A-subset S of M. This means the smallest A-submodule M', in the lattice of submodules of M, such that ${}_aS \subseteq {}_aM'$ for each $a \in A$.

Theorem 4.1. *Let M be an A-module or pointed A-overalgebra. We have*

1. *If $M' \leq M$ is a submodule or sub pointed overalgebra, respectively, then $A \ltimes M'$ is a subalgebra of $A \ltimes M$; and*
2. $\operatorname{Sub} M \cong \mathrm{I}_{\operatorname{Sub} A \ltimes M}[A, \top]$ *under the mapping $M' \mapsto A \ltimes M'$.*

Example 4.1. Let $\phi : M \to M'$ be a homomorphism of A-modules. We define $\operatorname{Ke} \phi$ by ${}_a(\operatorname{Ke} \phi) = \operatorname{Ke}({}_a\phi)$, and this can easily be shown to be a submodule of M, and to be a kernel of ϕ in the sense of the theory of additive categories.

Quotient objects. If M is a pointed A-overalgebra or A-overalgebra, a *congruence* of M is an $|A|$-tuple γ of equivalence relations ${}_a\gamma$ such that for each n-ary ω, each n-tuple $\mathbf{a}$ of elements of A, and each $\mathbf{m}$, $\mathbf{m}' \in {}_{\mathbf{a}}M$ such that $m_i \; {}_{a_i}\gamma \; m_i'$ for all i, we have $\omega_{\mathbf{a}}^M(\mathbf{m}) \; {}_{\omega(\mathbf{a})}\gamma \; \omega_{\mathbf{a}}^M(\mathbf{m}')$. If M is an A-module, then we impose the additional condition that each ${}_a\gamma$ be an abelian group congruence of ${}_aM$. The congruences of M are partially ordered in an obvious manner, and this gives rise to a complete lattice of congruences, $\operatorname{Con} M$.

If M and $\gamma \in \operatorname{Con} M$ are given, then there is a unique structure of A-module (pointed A-overalgebra, A-overalgebra) on the $|A|$-tuple of abelian groups (respectively, pointed sets, sets) ${}_aM/{}_a\gamma$, which we denote by M/γ, such that the natural maps $\operatorname{nat}{}_a\gamma$ form a homomorphism $\operatorname{nat}\gamma : M \to M/\gamma$ of A-modules (respectively, of pointed A-overalgebras, A-overalgebras).

Just as $\operatorname{Sub} M$ is isomorphic to a lattice of equivalence classes of one-one homomorphisms with codomain M, $\operatorname{Con} M$ is isomorphic to a lattice of equivalence classes of onto homomorphisms with domain M.

Theorem 4.2. *Let A be an algebra, and let M be an A-module, pointed A-overalgebra, or A-overalgebra.*

1. *If $\gamma \in \operatorname{Con} M$, then the binary relation $A \ltimes \gamma$, defined by $\langle a, m\rangle \; A \ltimes \gamma \; \langle a', m'\rangle$ iff $a = a'$ and $m \; {}_a\gamma \; m'$, is a congruence of $A \ltimes M$; and*
2. $\operatorname{Con} M \cong \mathrm{I}_{\operatorname{Con} A \ltimes M}[\bot, \ker \pi_M]$, *via the mapping $\gamma \mapsto A \ltimes \gamma$.*

Just as a congruence on an abelian group is determined by the equivalence class of the 0 element, a congruence γ on an A-module M is determined by the submodule M' such that each ${}_aM'$ is the ${}_a\gamma$-class of ${}_a0^M$.

Example 4.2. Let $\phi : M \to M'$ be a homomorphism of A-modules. We define $\operatorname{Co}\phi$ by ${}_a(\operatorname{Co}\phi) = \operatorname{Co}({}_a\phi)$. It is straightforward to prove that there is a unique structure of A-module on $\operatorname{Co}\phi$, making the natural maps from M' onto $\operatorname{Co}\phi$ a homomorphism, and that $\operatorname{Co}\phi$ is a cokernel of ϕ.

5. The Clone of Polynomials

Let A be an algebra in a variety $\mathbf{V}$ of algebras of type Ω. We define the *algebra of n-ary polynomials with coefficients in A (relative to $\mathbf{V}$)* to be the coproduct $A \coprod \mathcal{F}_{\mathbf{V}}\{\, x_1, \dots, x_n \,\}$. Of course, this algebra is defined only up to isomorphism, but we choose one such algebra, and denote it by Q_n. For each n, there is a distinguished one-one homomorphism from A to Q_n, and we will use the same notation ι_A for any of these homomorphisms. For $a \in A$, we will typically write a for $\iota_A(a)$ in our formulas.

If $\Pi \in Q_n$, and $\mathbf{a}$ is an n-tuple of elements of A, then we define $\Pi^A(\mathbf{a})$ to be the image of Π in A, under the unique homomorphism $\phi : Q_n \to A$ sending each $a \in A$ to a and each x_i to a_i. Such a homomorphism exists, and is unique, by the universal property of Q_n. Similarly, if B is an algebra in $\mathbf{V}$ with a distinguished homomorphism $f : A \to B$, and $\mathbf{b}$ is an n-tuple of elements of B, then we define $\Pi^{B,f}(\mathbf{b})$ to be the image in B of Π, under the unique homomorphism $\phi : Q_n \to B$ sending each $a \in A$ to $f(a)$, and each x_i to b_i.

We will write x for x_1 when talking about unary polynomials, just as we do with unary terms.

Theorem 5.1. *Let A be an algebra in a variety $\mathbf{V}$ of algebras of type Ω. We have*

1. *If we define $\operatorname{Pol}_n(A, \mathbf{V}) = |Q_n|$, $\pi_{i,n}^{\operatorname{Pol}(A,\mathbf{V})} = x_i$ and, given n, n', an n'-tuple $\mathbf{\Pi}$ of n-ary polynomials, and an n'-ary polynomial Π', $\Pi'\mathbf{\Pi} = (\Pi')^{Q_n,\iota_A}(\mathbf{\Pi})$, then $\operatorname{Pol}(A, \mathbf{V})$ is a clone;*
2. *The mappings $(\Phi^A_{\operatorname{Pol}})_n : \operatorname{Pol}_n(A, \mathbf{V}) \to \operatorname{Clo}_n |A|$, given by $\Pi \mapsto \Pi^A$, form a clone homomorphism;*
3. *if B is an algebra, provided with a homomorphism $f : A \to B$, the mappings $\Phi_n^{B,f}$: $\operatorname{Pol}_n(A, \mathbf{V}) \to \operatorname{Clo}_n |B|$, given by $\Pi \mapsto \Pi^{B,f}$, form a clone homomorphism; and*
4. *there is a natural clone homomorphism $\tilde{\Phi} : \operatorname{Clo}\Omega \to \operatorname{Pol}(A, \mathbf{V})$, such that $\Phi^{B,f}\tilde{\Phi} = \Phi^B$.*

Proof. (1), (2), and (3) are straightforward. For (4), $\tilde{\Phi}_n$ is simply the insertion of the relatively free algebra into the coproduct. □

Remark. $\tilde{\Phi}_n$ may not be one-one.

Polynomials and polynomial functions. We have defined a clone $\operatorname{Pol}(A, \mathbf{V})$ of polynomials with coefficients in A (relative to $\mathbf{V}$), and, given an algebra $B \in \mathbf{V}$ with a homomorphism $f : A \to B$, a clone homomorphism $\Phi^{B,f} : \operatorname{Pol}(A, \mathbf{V}) \to \operatorname{Clo}|B|$, which sends each n-ary polynomial Π to $\Pi^{B,f}$. The image of this clone homomorphism is a subclone of $\operatorname{Clo}|B|$, the

clone of polynomial functions in B with coefficients in A. As with terms and term functions, it is necessary to distinguish between these two clones.

Algebras of polynomials and free pointed overalgebras. Recall (example 2.1) our recipe for a pointed overalgebra free on an A-set. An n-tuple $\mathbf{a}$ of elements of A determines an A-set S with disjoint union $\{ x_1, \dots , x_n \}$, by the rule $x_i \in {}_{\bar{a}}S \iff a_i = \bar{a}$. Then if $\pi_{\mathbf{a}}$ is evaluation at $\mathbf{a}$, i.e., the homomorphism $\Pi \mapsto \Pi^A(\mathbf{a})$, we have $[\![\mathrm{Pol}_n(A, \mathbf{V}), \pi_{\mathbf{a}}, \iota_A]\!]$ a pointed A-overalgebra, totally in $\mathbf{V}$ and free on S. We will use this fact when we use the algebra of unary polynomials in our construction of the enveloping ringoid, and just below in the definition of the A-operation Π^P for P a pointed A-overalgebra.

Polynomials and pointed overalgebras. Given $\Pi \in \mathrm{Pol}_n(A, \mathbf{V})$, $P \in \mathbf{Pnt}[A, \mathbf{V}]$, $\mathbf{a} \in A^n$, and $\mathbf{p} \in {}_{\mathbf{a}}P$, we define $\Pi^P_{\mathbf{a}}(\mathbf{p})$ to be the image of Π in P, under the unique homomorphism from $[\![\mathrm{Pol}_n(A, \mathbf{V}), \pi_{\mathbf{a}}, \iota_A]\!]$ to P sending each x_i to p_i.

Theorem 5.2. *Let A be an algebra, and $P \in \mathbf{Pnt}(A, \mathbf{V})$. We have*

1. *The $\mathbb{N}$-tuple of mappings sending n-ary polynomials Π to $\langle \Pi^A, \Pi^P \rangle$ form a clone homomorphism $\Phi^P_{\mathrm{Pol}} : \mathrm{Pol}(A, \mathbf{V}) \to \mathrm{Clo}\, M$, such that $\Lambda_P \Phi^P_{\mathrm{Pol}} = \Phi^A_{\mathrm{Pol}}$ and $\Phi^P_{\mathrm{Pol}} \tilde{\Phi} = \Phi^P$;*
2. *for each n-ary polynomial Π, and n-tuple $\langle a_1, p_1 \rangle, \dots, \langle a_n, p_n \rangle \in A \ltimes P$, we have $\Pi^{\langle A \ltimes P, \iota_P \rangle}(\langle a_1, p_1 \rangle, \dots , \langle a_n, p_n \rangle) = \langle \Pi^A(\mathbf{a}), \Pi^P_{\mathbf{a}}(\mathbf{p}) \rangle$;*
3. *if P' is another pointed A-overalgebra, $\phi : P \to P'$ a homomorphism, and $\Pi \in \mathrm{Pol}_n(A, \mathbf{V})$, then ϕ sends Π^P to $\Pi^{P'}$; and*
4. *if $B \in \mathbf{V}$, $\pi : B \to A$, and $\iota : A \to B$ are such that $\pi\iota = 1_A$, then $\Pi^{[\![B,\pi,\iota]\!]}_{\mathbf{a}}(\mathbf{b}) = \Pi^{B,\iota}(\mathbf{b})$.*

Proof. All parts are straightforward, if not obvious, except perhaps (3), the key to proving which is to know that if $\omega \in \mathrm{Clo}_{n'} \Omega$, then ϕ sends $\tilde{\Phi}(\omega)^P = \omega^P$ to $\omega^{P'} = \tilde{\Phi}(\omega)^{P'}$ because it is a homomorphism, and then to use theorem 3.2. □

Polynomials and A-modules. There is a natural forgetful functor $\mathcal{U} : \mathbf{Ab}[A, \mathbf{V}] \to \mathbf{Pnt}[A, \mathbf{V}]$, which takes an A-module M to the pointed A-overalgebra $\mathcal{U}M$ defined by ${}_a\mathcal{U}M = {}_aM$, ${}_a*^{\mathcal{U}M} = {}_a0^M$. If $\Pi \in \mathrm{Pol}_n(A, \mathbf{V})$ and $\mathbf{a} \in A^n$, we define $\Pi^M_{\mathbf{a}} = \Pi^{\mathcal{U}M}_{\mathbf{a}}$.

Theorem 5.3. *Let M be an A-module totally in $\mathbf{V}$, $\Pi \in \mathrm{Pol}_n(A, \mathbf{V})$, and $\mathbf{a} \in A^n$. Then $\Pi^M_{\mathbf{a}}$ is a group homomorphism and $\langle \Pi^A, \Pi^M \rangle \in \mathrm{Clo}_n M$.*

Proof. We use the fact that $\mathrm{Pol}_n(A, \mathbf{V})$ is generated by the projections and the constants $a \in A$. Certainly the conclusion is true for Π one of these generators. Suppose it is true for the components of $\mathbf{\Pi} \in \mathrm{Pol}_n(A, \mathbf{V})^{n'}$, and let $\omega \in \Omega_{n'}$. We have $(\Phi^M_{\mathrm{Pol}})_{n'}(\omega) = \langle \omega^A, \omega^M \rangle \in \mathrm{Clo}_{n'} M$, and $\langle \Pi^A_i, \Pi^M_i \rangle \in \mathrm{Clo}_n M$ for all i. Thus, by theorem 3.1, $\tilde{\Phi}(\omega)^M \circ_{\mathbf{\Pi}^A} \mathbf{\Pi}^M$ is an A-operation on M over $\tilde{\Phi}(\omega)\mathbf{\Pi}$. But, we have

$$\begin{aligned}
(\tilde{\Phi}(\omega)^M \circ_{\mathbf{\Pi}^A} \mathbf{\Pi}^M)_{\mathbf{a}}(\mathbf{m}) &= (\tilde{\Phi}(\omega)^{\mathcal{U}M} \circ_{\mathbf{\Pi}^A} \mathbf{\Pi}^{\mathcal{U}M})_{\mathbf{a}}(\mathbf{m}) \\
&= (\tilde{\Phi}(\omega)\mathbf{\Pi})^{\mathcal{U}M}_{\mathbf{a}}(\mathbf{m}) \\
&= (\tilde{\Phi}(\omega)\mathbf{\Pi})^{M}_{\mathbf{a}}(\mathbf{m})
\end{aligned}$$

for every $\mathbf{m} \in {}_{\mathbf{a}}M$, by definition and by the fact that $\Phi^{\mathcal{U}M}_{\mathrm{Pol}}$ is a clone homomorphism. Thus, the conclusion is true for $\Pi = \tilde{\Phi}(\omega)\mathbf{\Pi}$. $\square$

.

6. Z_M

Let A be a set, and M an A-tuple of abelian groups. If w is an n-ary operation on A, W is an n-ary A-operation on M over w, $\mathbf{a}$ is an n-tuple of elements of A, and $1 \le i \le n$, then we denote by $W_{\mathbf{a},i}$ the abelian group homomorphism from ${}_{a_i}M$ to ${}_{w(\mathbf{a})}M$ defined by $m \mapsto W_{\mathbf{a}}({}_{a_1}0, \dots, {}_{a_{i-1}}0, m, {}_{a_{i+1}}0, \dots, {}_{a_n}0)$.

Theorem 6.1. *Let A be a set with an n-ary operation w, and M and M' A-tuples of abelian groups. Let W be an A-operation over w on M, and W' an A-operation over w on M'.*

1. *If $M = M'$, then $W = W'$ iff $W_{\mathbf{a},i} = W'_{\mathbf{a},i}$ for all $\mathbf{a}$ and i.*
2. *If ϕ is an A-tuple of homomorphisms ${}_a\phi : {}_aM \to {}_aM'$, then ϕ sends W to W' iff for all i such that $1 \le i \le n$, $\mathbf{a} \in A^n$, and $m \in {}_{a_i}M$, we have*
$$\phi_{w(\mathbf{a})}(W_{\mathbf{a},i}(m)) = W'_{\mathbf{a},i}(\phi_{a_i}(m)).$$

If A is an algebra and M is an A-module, then the $\omega^M_{\mathbf{a},i}$ are of particular interest. We denote by Z_M the smallest subringoid of $\mathrm{End}(M)$ containing all of the $\omega^M_{\mathbf{a},i}$.

Theorem 6.2. *Let M be an A-module. We have*

1. *In order for an A-tuple of subgroups ${}_aM' \subseteq {}_aM$ to be a submodule, it is necessary and sufficient that for each n-ary operation symbol ω, each n-tuple $\mathbf{a}$ of elements of A, and each i such that $1 \le i \le n$, we have $\omega_{\mathbf{a},i}({}_{a_i}M') \subseteq {}_{\omega(\mathbf{a})}M'$; and*
2. *if M' is the submodule of M generated by an A-subset S, then for each $a \in A$, ${}_aM'$ is the subgroup of ${}_aM$ generated by elements of the form rm, where $r \in {}_a(Z_M)_b$ for some $b \in A$, and $m \in {}_bS$.*

7. Modulization of a Pointed Overalgebra

Recall (§5) that we denote by $\mathcal{U}$ the natural forgetful functor from $\mathbf{Ab}[A]$ to $\mathbf{Pnt}[A]$, which assigns to each A-module M the pointed overalgebra $\mathcal{U}M$ such that ${}_a(\mathcal{U}M) = {}_aM$ and ${}_a*^{\mathcal{U}M} = {}_a0^M$. For arrows, $\mathcal{U}M$ sends a homomorphism $\phi : M \to M'$, consisting of an A-tuple of abelian group homomorphisms ${}_a\phi : {}_aM \to {}_aM'$, to the same A-tuple of functions, which are also pointed set homomorphisms and comprise a pointed overalgebra homomorphism. Note that if $\mathbf{V}$ is a variety of algebras, $\mathcal{U}$ sends $\mathbf{Ab}[A, \mathbf{V}]$ into $\mathbf{Pnt}[A, \mathbf{V}]$ and we can view $\mathcal{U}$ as a functor from $\mathbf{Ab}[A, \mathbf{V}]$ to $\mathbf{Pnt}[A, \mathbf{V}]$.

In this section, we will construct universal arrows to all of these functors. In §8, we will use the resulting left adjoint functors to construct enveloping ringoids.

We will call the left adjoint to $\mathcal{U} : \mathbf{Ab}[A] \to \mathbf{Pnt}[A]$ the functor of *modulization* and denote it by $\mathcal{M}$. However, we will begin by defining an easier functor $\hat{\mathcal{M}} : \mathbf{Pnt}[A] \to \mathbf{Ab}[A]$. If P is a pointed A-overalgebra, then for each $a \in A$, ${}_a(\hat{\mathcal{M}}P)$ will be the free abelian group

on ${}_aP$. We will identify each $p \in {}_aP$ with the corresponding generator of ${}_a(\hat{\mathcal{M}}P)$. If ω is an n-ary operation symbol, and $\mathbf{a}$ an n-tuple of elements of A, then we will define each $\omega_{\mathbf{a},i}^{\hat{\mathcal{M}}(P)}$ on the generators of ${}_{a_i}\hat{\mathcal{M}}(P)$ by the equation

$$\omega_{\mathbf{a},i}^{\hat{\mathcal{M}}P}(p) = \omega^P({}_{a_1}*, \ldots , {}_{a_{i-1}}*, p, {}_{a_{i+1}}*, \ldots , {}_{a_n}*);$$

the $\omega_{\mathbf{a},i}^{\hat{\mathcal{M}}P}$ then determine the A-operations $\omega^{\hat{\mathcal{M}}P}$.

In order to obtain the left adjoint functor $\mathcal{M}$, we must have a unit natural transformation $\eta : 1_{\mathbf{Pnt}[A]} \overset{\bullet}{\to} \mathcal{U}\mathcal{M}$. We have a natural definition for functions ${}_a(\hat{\eta}P) : {}_aP \to {}_a(\mathcal{U}\hat{\mathcal{M}}P)$; we simply send each $p \in {}_aP$ to itself, viewed as an element of ${}_a(\hat{\mathcal{M}}P)$. However, the A-tuple of functions $\hat{\eta}P$ is not a homomorphism of pointed A-overalgebras, because it does not send each ${}_a*$ to ${}_a0$ and does not satisfy the equations

$${}_{\omega(\mathbf{a})}(\hat{\eta}P)(\omega_{\mathbf{a}}^P(\mathbf{p})) = \omega_{\mathbf{a}}^{\mathcal{U}\hat{\mathcal{M}}P}({}_{a_1}(\hat{\eta}P)(p_1), \ldots , {}_{a_n}(\hat{\eta}P)(p_n))$$

for each n-ary operation symbol ω, each n-tuple $\mathbf{a}$ of elements of A, and each $\mathbf{p} \in {}_{\mathbf{a}}P$.

Accordingly, we define $\mathcal{M}P$ to be the quotient of $\hat{\mathcal{M}}P$ by the smallest submodule $\mathcal{K}P$ such that the composite A-function ηP of the A-function $\hat{\eta}P$ and the natural homomorphism $\nu : \hat{\mathcal{M}}P \to \mathcal{M}P$ yields a homomorphism of pointed A-overalgebras. Let $\mathcal{S}P$ be the smallest A-subset of $\hat{\mathcal{M}}P$ such that each ${}_a\mathcal{S}P$ contains ${}_a*$, and such that for each n-ary operation symbol ω, each n-tuple $\mathbf{a}$ of elements of A, and each $\mathbf{p} \in {}_{\mathbf{a}}P$, ${}_{\omega(\mathbf{a})}\mathcal{S}P$ contains $\omega_{\mathbf{a}}^P(\mathbf{p}) - \sum_{i=1}^n \omega_{\mathbf{a},i}^{\hat{\mathcal{M}}P}(p_i) = \omega_{\mathbf{a}}^P(\mathbf{p}) - \omega_{\mathbf{a}}^{\hat{\mathcal{M}}P}(\mathbf{p})$. Then $\mathcal{K}P$ is the submodule of $\hat{\mathcal{M}}P$ generated by the A-set $\mathcal{S}P$.

Theorem 7.1. *For all $P \in \mathbf{Pnt}[A]$, ηP is a homomorphism of pointed A-overalgebras, and $\langle \mathcal{M}P, \eta P\rangle$ is a universal arrow from P to $\mathcal{U}$.*

Proof. Let $\omega \in \Omega_n$, $\mathbf{a} \in A^n$, and $\mathbf{p} \in {}_{\mathbf{a}}P$. For each $a \in A$, ${}_a\eta P$ preserves the basepoint, because ${}_a*^P \in {}_a\mathcal{S}P$, so that

$${}_a\eta P({}_a*^P) = \nu({}_a*^P) = {}_a0^{\mathcal{M}P}.$$

Also, because $\omega_{\mathbf{a}}^P(\mathbf{p}) - \omega_{\mathbf{a}}^{\mathcal{M}P} \in {}_{\omega(\mathbf{a})}\mathcal{S}P$, we have

$$\begin{aligned}
\omega_{\mathbf{a}}^{\mathcal{U}\mathcal{M}P}({}_{a_1}\eta P(p_1), \ldots , {}_{a_n}\eta P(p_n)) &= \omega_{\mathbf{a}}^{\mathcal{M}P}({}_{a_1}\nu(p_1), \ldots , {}_{a_n}\nu(p_n)) \\
&= {}_{\omega(\mathbf{a})}\nu\left(\omega_{\mathbf{a}}^{\hat{\mathcal{M}}P}(\mathbf{p})\right) \\
&= {}_{\omega(\mathbf{a})}\nu\left(\omega_{\mathbf{a}}^P(\mathbf{p})\right) \\
&= ({}_{\omega(\mathbf{a})}\nu)({}_{\omega(\mathbf{a})}\hat{\eta}P)(\omega_{\mathbf{a}}^P(\mathbf{p})) \\
&= {}_{\omega(\mathbf{a})}\eta P(\omega_{\mathbf{a}}^P(\mathbf{p})).
\end{aligned}$$

Thus, ηP is a homomorphism.

Now let $M \in \mathbf{Ab}[A]$, and $\zeta : P \to \mathcal{U}M$. We must show that there is a unique A-module homomorphism $\xi : \mathcal{M}P \to M$ such that $\zeta = (\mathcal{U}\xi)(\eta P)$. By the construction of $\hat{\mathcal{M}}P$, there

is a unique A-module homomorphism $\hat{\xi} : \hat{\mathcal{M}}P \to M$ such that $\zeta = (\mathcal{U}\hat{\xi})(\hat{\eta}P)$; this shows that ξ is unique if it exists, because then $\xi\nu = \hat{\xi}$, and ν is an epimorphism.

To show ξ exists, we must show that for each a, ${}_a\hat{\xi}$ is zero on ${}_a\mathcal{S}P$. But, we have ${}_a\hat{\xi}({}_a{*}^P) = {}_a0^M$, because ${}_a\zeta({}_a{*}^P) = {}_a{*}^{\mathcal{U}M} = {}_a0^M$, and if ω is an n-ary operation symbol, $\mathbf{a} \in A^n$, and $\mathbf{p} \in {}_{\mathbf{a}}P$, we have

$$\begin{aligned}
{}_{\omega(\mathbf{a})}\hat{\xi}\left(\omega_{\mathbf{a}}^P(\mathbf{p}) - \sum_{i=1}^n \omega_{\mathbf{a},i}^{\hat{\mathcal{M}}P}(p_i)\right) &= {}_{\omega(\mathbf{a})}\hat{\xi}(\omega_{\mathbf{a}}^P(\mathbf{p})) - \sum_i {}_{\omega(a)}\hat{\xi}(\omega_{\mathbf{a},i}^{\hat{\mathcal{M}}P}(p_i)) \\
&= \omega_{\mathbf{a}}^M({}_{\mathbf{a}}\hat{\xi}(\mathbf{p})) - \sum_{i=1}^n \omega_{\mathbf{a},i}^M({}_{a_i}\hat{\xi}(p_i)) \\
&= \omega_{\mathbf{a}}^M({}_{\mathbf{a}}\zeta(\mathbf{p})) - \sum_{i=1}^n \omega_{\mathbf{a},i}^M({}_{a_i}\zeta(p_i)) \\
&= {}_{\omega(\mathbf{a})}0.
\end{aligned}$$

□

It is also obvious from the construction that we have

Theorem 7.2. *Let P be a pointed A-overalgebra, and $a \in A$. Then every element of ${}_a(\mathcal{M}P)$ is a $\mathbb{Z}$-linear combination of images of elements of ${}_aP$ under ${}_a(\eta P)$.*

If $\mathbf{V}$ is a variety of algebras to which A belongs, we also want a left adjoint functor to the restriction of $\mathcal{U}$ to $\mathbf{Ab}[A, \mathbf{V}]$. It suffices to have $\mathcal{M}P$ always be totally in $\mathbf{V}$ if P is; then the restriction of $\mathcal{M}$ to $\mathbf{Pnt}[A, \mathbf{V}]$ will be the desired left adjoint.

Theorem 7.3. *We have*

1. *$\mathcal{M}P$ satisfies any identity satisfied by P.*
2. *If P is a pointed A-overalgebra totally in $\mathbf{V}$, then $\mathcal{M}P$ is totally in $\mathbf{V}$.*
3. *If P is totally in $\mathbf{V}$, then $\langle \mathcal{M}P, \eta P\rangle$ is a universal arrow from P to the restriction of $\mathcal{U}$ to $\mathbf{Ab}[A, \mathbf{V}]$.*

Proof. (3) follows from (2), which follows from (1). To prove (1), first we prove that for each n-ary term t, n-tuple $\mathbf{a}$ of elements of A, i such that $1 \le i \le n$, and $p \in {}_{a_i}P$, we have

$$t_{\mathbf{a},i}^{\hat{\mathcal{M}}P}(p) - t_{\mathbf{a}}^P({}_{a_1}{*}, \dots, {}_{a_{i-1}}{*}, p, {}_{a_{i+1}}{*}, \dots, {}_{a_n}{*}) \in {}_{t^A(\mathbf{a})}\mathcal{K}P; \tag{7.1}$$

we will use the fact that $\mathrm{Clo}_n\,\Omega$ is generated by the x_i. Let $j \in \{1, \dots, n\}$. We have

$$\begin{aligned}
&(x_j)_{\mathbf{a},i}^{\hat{\mathcal{M}}P}(p) - (x_j)_{\mathbf{a}}^P({}_{a_1}{*}, \dots, {}_{a_{i-1}}{*}, p, {}_{a_{i+1}}{*}, \dots, {}_{a_n}{*}) \\
&\qquad = \begin{cases} p - p, & \text{if } i = j\text{, and} \\ -{}_{a_j}{*}, & \text{if } i \ne j \end{cases} \\
&\qquad \in {}_{a_j}\mathcal{K}P.
\end{aligned}$$

Now, suppose ω is an m-ary operation symbol, and (7.1) is true for n-ary terms $s_1, \dots, s_m$. Then we will show it is true for the n-ary term $\omega\mathbf{s}$. We have

$$\begin{aligned}
(\omega\mathbf{s})^{\hat{\mathcal{M}}P}_{\mathbf{a},i}(p) &= \omega^{\hat{\mathcal{M}}P}_{\mathbf{s}(\mathbf{a})}\left((s_1)^{\hat{\mathcal{M}}P}_{\mathbf{a},i}(p), \dots, (s_m)^{\hat{\mathcal{M}}P}_{\mathbf{a},i}(p)\right)\\
&= \sum_{j=1}^{m} \omega^{\hat{\mathcal{M}}P}_{\mathbf{s}(\mathbf{a}),j}\left((s_j)^{\hat{M}P}_{\mathbf{a},i}(p)\right)\\
&= \sum_{j=1}^{m} \omega^{\hat{\mathcal{M}}P}_{\mathbf{s}(\mathbf{a}),j}\left((s_j)^P_{\mathbf{a}}({}_{a_1}*, \dots, {}_{a_{i-1}}*, p, {}_{a_{i+1}}*, \dots, {}_{a_n}*) + k_j\right)\\
&= \left(\sum_{j=1}^{m} \omega^{\hat{\mathcal{M}}P}_{\mathbf{s}(\mathbf{a}),j}\left((s_j)^P_{\mathbf{a}}({}_{a_1}*, \dots, {}_{a_{i-1}}*, p, {}_{a_{i+1}}*, \dots, {}_{a_n}*)\right)\right) + k\\
&= \omega^P_{\mathbf{s}(\mathbf{a})}\left(\dots, (s_j)^P_{\mathbf{a}}({}_{a_1}*, \dots, {}_{a_{i-1}}*, p, {}_{a_{i+1}}*, \dots, {}_{a_n}*), \dots\right) + k'\\
&= (\omega\mathbf{s})^P_{\mathbf{a}}({}_{a_1}*, \dots, {}_{a_{i-1}}*, p, {}_{a_{i+1}}*, \dots, {}_{a_n}*) + k'
\end{aligned}$$

where each $k_j \in {}_{s_j(\mathbf{a})}\mathcal{K}P$, and k, $k' \in {}_{\omega\mathbf{s}(\mathbf{a})}\mathcal{K}P$

Now, each A-operation $t^{\hat{\mathcal{M}}P}$ is determined by the actions of the $t^{\hat{\mathcal{M}}P}_{\mathbf{a},i}$ on elements of ${}_{a_i}P$. If t and t' are n-ary terms such that $t^P = (t')^P$, then $t^A = (t')^A$ because $\pi_P : A \ltimes P \to P$ is onto, and (7.1) implies that values of $t^{\hat{\mathcal{M}}P}$ and $(t')^{\hat{\mathcal{M}}P}$ differ by elements of the submodule $\mathcal{K}P$. This implies that $t^{\mathcal{M}P} = (t')^{\mathcal{M}P}$. In other words, $\mathcal{M}P$ satisfies any identity satisfied by P. $\square$

8. Enveloping Ringoids

In this section, we will use the modulization functor $\mathcal{M} : \mathbf{Pnt}[A, \mathbf{V}] \to \mathbf{Ab}[A, \mathbf{V}]$ to construct the enveloping ringoid $\mathbb{Z}[A, \mathbf{V}]$ of an algebra A in a variety $\mathbf{V}$.

If A is an algebra in a variety $\mathbf{V}$, we will abbreviate $\mathrm{Pol}_1(A, \mathbf{V})$ by U, and continue to denote the insertion of A into U by ι_A. If $b \in A$, we will write U_b for the pointed overalgebra $[\![U, \pi_b, \iota_A]\!]$, where $\pi_b : U \to A$ is the homomorphism given by $u \mapsto u(b)$. If $a \in A$, we will write ${}_aU_b$ for ${}_a(U_b)$, and if $\mathbf{a}$ is an n-tuple of elements of A, we will write ${}_{\mathbf{a}}U_b$ for ${}_{a_1}U_b \times \dots \times {}_{a_n}U_b$.

Now, as discussed previously in §5, U_b is a free pointed A-overalgebra, totally in $\mathbf{V}$, on one generator, $x \in {}_bU_b$. In other words, U_b is free on the A-subset S such that ${}_bS = \{\, x \,\}$ and ${}_aS = \varphi$ for $a \neq b$. From the fact that $\mathcal{M}$ is left adjoint to $\mathcal{U}$, it follows that $\mathcal{M}U_b \in \mathbf{Ab}[A, \mathbf{V}]$ is free on ${}_b(\eta U_b)(x)$, where η is the unit natural transformation, described in §7, of the adjunction between $\mathcal{M}$ and $\mathcal{U}$.

Let P be a pointed A-overalgebra totally in $\mathbf{V}$. Since U_b is free on $x \in {}_bU_b$, if $p \in {}_bP$, then there is a unique homomorphism $\psi^P_{b,p} : U_b \to P$ taking x to p. We will denote this homomorphism by $\psi_{b,p}$ when there is no confusion. Recall that by the definition in §5, if $p \in {}_bP$, and u is a unary polynomial relative to $\mathbf{V}$, then ${}_{u(b)}(\psi_{b,p})(u) = u^P_{\langle b\rangle}(p)$.

Similarly, suppose M is an A-module totally in $\mathbf{V}$. Since $\mathcal{M}U_b$ is free on ${}_b(\eta U_b)(x)$, if $m \in {}_bM$, there is a unique homomorphism $\theta_{b,m} = \theta^M_{b,m} : \mathcal{M}U_b \to M$ taking ${}_b(\eta U_b)(x)$ to m. We have $(\theta^M_{b,m})(\eta U_b) = \psi^{\mathcal{U}M}_{b,m}$.

We define ${}_aZ_b = {}_a(\mathcal{M}U_b)$ for $a,\, b \in A$. If $u \in U$ and $b \in A$, we also define $(u)_b$ to mean ${}_{u(b)}(\eta U_b)(u)$. That is, $(u)_b$ is the equivalence class of u in the modulization of U_b. Let M be an A-module totally in $\mathbf{V}$. If $z \in {}_aZ_b$, and $m \in {}_bM$, then we define zm to be ${}_a(\theta_{b,m})(z)$. Given $z \in {}_aZ_b$ and $z' \in {}_bZ_c$, we define zz' to be the result of the action of z on $z' \in {}_b(\mathcal{M}U_c)$.

Theorem 8.1. *Let A be an algebra in $\mathbf{V}$, and let M be an A-module totally in $\mathbf{V}$. Let Z be as just defined. We have*

1. $(x)_bm = m$;
2. *if* $z \in {}_aZ_b$, *then* $z(x)_b = z$ *and* $(x)_az = z$;
3. *the action* $\langle z, m\rangle \mapsto zm$ *is bilinear;*
4. *the multiplication* $\langle z, z'\rangle \mapsto zz'$ *is bilinear;*
5. *if* $u \in U$, *and* $m \in {}_bM$, *then* $(u)_bm = u^M_{\langle b\rangle}(m)$;
6. *if* $u,\, v \in U$, $(v)_{u(b)}(u)_b = (vu)_b$;
7. *the action of* Z *on* M *is associative, i.e.,* $(z'z)m = z'(zm)$;
8. *multiplication in* Z *is associative;*
9. Z *is a ringoid, with* $1^Z_b = (x)_b$ *and* ${}_a0^Z_b = (a)_b$; *and*
10. *the* $|A|$*-tuple of abelian groups* M, *with the defined action by* Z, *form a left* Z*-module.*

Now let M' be another A-module totally in $\mathbf{V}$, and $\phi : M \to M'$ a homomorphism. We have

11. *If* $z \in {}_aZ_b$ *and* $m \in {}_bM$, *then* $z({}_b\phi(m)) = {}_a\phi(zm)$.

Proof. (1): This is clear from the definition of the action of Z on M.

(2) That $z(x)_b = z$ follows from the fact that $\theta_{b,(x)_b} = 1_{\mathcal{M}U_b}$. That $(x)_az = z$ follows from (1).

(3): Let $m \in {}_bM$ and $z \in {}_aZ_b$. If $m = n_1m_1 + n_2m_2$, then $\theta_{b,m} = n_1\theta_{b,m_1} + n_2\theta_{b,m_2}$. Thus, $zm = {}_a(\theta_{b,m})(z) = n_{1a}(\theta_{b,m_1})(z) + n_{2a}(\theta_{b,m_2})(z) = zm_1 + zm_2$. If $z = n_1z_1 + n_1z_2$, then ${}_a(\theta_{b,m})(z) = n_{1a}(\theta_{b,m})(z_1) + n_{2a}(\theta_{b,m})(z_2) = z_1m + z_2m$.

(4): Follows from (3).

(5): We have

$$\begin{aligned}
(u)_bm &= {}_{u(b)}(\theta_{b,m})((u)_b) \\
&= {}_{u(b)}(\theta_{b,m})({}_{u(b)}\eta U_b)(u)) \\
&= {}_{u(b)}(\psi_{b,m})(u) \\
&= u^{\mathcal{U}M}_{\langle b\rangle}(m) \\
&= u^M_{\langle b\rangle}(m),
\end{aligned}$$

by the definition of $u^M_{\langle b\rangle}(m)$ from §5.

(6): Using (5), we have

$$\begin{aligned}(v)_{u(b)}(u)_b &= v^{Z_b}_{\langle u(b)\rangle}((u)_b)\\ &= v^{Z_b}_{\langle u(b)\rangle}({}_{u(b)}(\eta U_b)(u))\\ &= {}_{v(u(b))}(\eta U_b)(v^{U_b}_{\langle u(b)\rangle}(u))\\ &= {}_{v(u(b))}(\eta U_b)(v^{U,\iota_A}(u))\\ &= {}_{v(u(b))}(\eta U_b)(vu)\\ &= (vu)_b.\end{aligned}$$

(7): From theorem 7.2, z' and z are linear combinations respectively of elements of form $(u')_{b'}$ and $(u)_b$ where $b' = u(b)$. By (3), the statement to be proved then reduces to the same statement for $z = (u')_{u(b)}$ and $z = (u)_b$. But this follows from (6), (5), and the associativity of action for unary polynomials (theorem 5.2(1)).

(8): Follows from (7).

(9): Follows from (8), (4), and (2), and the fact that by definition, $(a)_b$ is the zero element of ${}_aZ_b$, because it is the image under the pointed A-overalgebra homomorphism ${}_a(\eta U_b)$ of ${}_a*^{U_b}$.

(10): Follows from (7), (3), and (1).

(11): Follows from the fact that $\theta_{b,b\phi(m)} = \phi\theta_{b,m}$. □

Z is the *enveloping ringoid of A with respect to* $\mathbf{V}$. More formally, we denote this ringoid by $\mathbb{Z}[A,\mathbf{V}]$. However, we will continue to abbreviate it by Z in what follows, when no confusion is likely.

9. The Isomorphism of $\mathbb{Z}[A,\mathbf{V}]$-Mod and $\mathbf{Ab}[A,\mathbf{V}]$

From theorem 8.1 (10) and (11), the construction of the left Z-module structure on M is a functor $\mathcal{G} : \mathbf{Ab}[A,\mathbf{V}] \to Z\text{-}\mathbf{Mod}$, where we define $\mathcal{G}\zeta = \zeta$ for $\zeta : M_1 \to M_2$.

Given a left Z-module M, we place an A-module structure $\mathcal{H}M$ on the A-tuple of abelian groups ${}_aM$ by defining

$$\omega^{\mathcal{H}M}_{\mathbf{a},i}(m) = \big(\omega^U(a_1,\dots,a_{i-1},x,a_{i+1},\dots,a_n)\big)_{a_i}(m),$$

for each n-ary operation symbol ω, n-tuple $\mathbf{a}$, $i \in \{1,\dots,n\}$, and $m \in {}_{a_i}M$.

Lemma 9.1. *If ω is an n-ary operation symbol, $\mathbf{a}$ is an n-tuple of elements of A, $1 \le i \le n$, and $z \in {}_{a_i}Z_b$, then $\omega^{Z_b}_{\mathbf{a},i}(z) = \big(\omega^U(a_1,\dots,a_{i-1},x,a_{i+1},\dots,a_n)\big)_{a_i} z$.*

Proof. We have

$$\begin{aligned}
\omega^{Z_b}_{\mathbf{a},i}(z) &= \omega^{Z_b}_{\mathbf{a},i}({}_{a_i}\theta_{a_i,z}(1_{a_i})) \\
&= {}_{\omega(\mathbf{a})}\theta_{a_i,z}(\omega^{Z_{a_i}}_{\mathbf{a},i}(1_{a_i})) \\
&= \Big(\omega^{Z_{a_i}}_{\mathbf{a}}({}_{a_1}0, \dots, {}_{a_{i-1}}0, x, {}_{a_{i+1}}0, \dots, {}_{a_n}0)\Big)\, z \\
&= \Big({}_{\omega(\mathbf{a})}(\eta U_{a_i})\omega^{U_{a_i}}_{\mathbf{a}}({}_{a_1}*, \dots, {}_{a_{i-1}}*, x, {}_{a_{i+1}}*, \dots, {}_{a_n}*)\Big)\, z \\
&= \big(\omega^U(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)\big)_{a_i}\, z.
\end{aligned}$$

$\square$

Lemma 9.2. *If t is an n-ary term, $\mathbf{a}$ an n-tuple of elements of A, i a number such that $1 \le i \le n$, and $m \in {}_{a_i}M$, then*

$$t^{\mathcal{H}M}_{\mathbf{a},i}(m) = \big(t^U(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)\big)_b\, m.$$

Proof. Uses the fact that $\mathrm{Clo}_n\,\Omega$ is generated by the x_i. For $t = x_j$, we have

$$t^{\mathcal{H}M}_{\mathbf{a},i}(m) = \begin{cases} {}_{a_j}0, & \text{if } i \ne j, \text{ and} \\ m, & \text{if } i = j. \end{cases}$$

and

$$\begin{aligned}
&(t^U(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n))_{a_i} m \\
&\qquad = \begin{cases} (a_j)_{a_i} m, & \text{if } i \ne j, \text{ and} \\ (x)_{a_i} m, & \text{if } i = j. \end{cases} \\
&\qquad = \begin{cases} {}_{a_j}0, & \text{if } i \ne j, \text{ and} \\ 1^Z_{a_i} m, & \text{if } i = j. \end{cases}
\end{aligned}$$

Thus, the statement is true in that case.

Now let ω be an k-ary operation symbol, and let the statement be true for a k-tuple $\mathbf{s}$ of n-ary terms. That is, if for each j we write u_j for $s^U_j(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$, we will

be assuming $(s_j)^{\mathcal{H}M}_{\mathbf{a},i}(m) = (u_j)_{a_i} m$ for all m. We have

$$\begin{aligned}
((\omega\mathbf{s})^U(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n))_{a_i} m& \\
&= \left(\omega^U(\dots, u_j, \dots)\right)_{a_i} m \\
&= \left({}_{\omega\mathbf{s}(\mathbf{a})}(\eta U_{a_i})(\omega^{U_{a_i}}_{\mathbf{s}(\mathbf{a})}(\dots, u_j, \dots))\right) m \\
&= \left(\omega^{Z_{a_i}}_{\mathbf{s}(\mathbf{a})}(\dots, (u_j)_{a_i}, \dots,)\right) m \\
&= \left(\sum_j \omega^{Z_{a_i}}_{\mathbf{s}(\mathbf{a}),j}(u_j)_{a_i}\right) m \\
&= \left(\sum_j \left(\omega^U(s_1(\mathbf{a}), \dots, s_{j-1}(\mathbf{a}), x, s_{j+1}(\mathbf{a}), \dots, s_k(\mathbf{a}))\right)_{s_j(\mathbf{a})} (u_j)_{a_i}\right) m \\
&= \sum_j \left(\omega^U(s_1(\mathbf{a}), \dots, s_{j-1}(\mathbf{a}), x, s_{j+1}(\mathbf{a}), \dots, s_k(\mathbf{a}))\right)_{s_j(\mathbf{a})} ((u_j)_{a_i} m) \\
&= \sum_j \omega^{\mathcal{H}M}_{\mathbf{s}(\mathbf{a}),j}((u_j)_{a_i} m) \\
&= \omega^{\mathcal{H}M}_{\mathbf{s}(\mathbf{a})}\left(\dots, (s_j)^{\mathcal{H}M}_{\mathbf{a},i}(m), \dots\right) \\
&= (\omega\mathbf{s})^{\mathcal{H}M}_{\mathbf{s}(\mathbf{a}),i}(m).
\end{aligned}$$

□

Theorem 9.3. *$\mathcal{H}$ is a functor from Z-**Mod** to $\mathbf{Ab}[A, \mathbf{V}]$, where for a homomorphism ϕ, $\mathcal{H}\phi = \phi$.*

Proof. Let M and M' be left Z-modules and $\phi : M \to M'$ a homomorphism. To prove that $\mathcal{H}$ is functorial, we must show that if ω is an n-ary operation symbol, $\mathbf{a} \in A^n$, $1 \le i \le n$, and $m \in {}_{a_i}M$, then

$$\omega^{\mathcal{H}M}_{\mathbf{a},i}({}_{a_i}\phi(m)) = {}_{\omega(\mathbf{a})}\phi(\omega^{\mathcal{H}M}_{\mathbf{a},i}(m)).$$

Let us write m' for ${}_{a_i}\phi(m)$. We have

$$\begin{aligned}
\omega^{\mathcal{H}M}_{\mathbf{a},i}({}_{a_i}\phi(m)) &= \left(\omega^U(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)\right)_{a_i} (m') \\
&= \left({}_{\omega(\mathbf{a})}(\eta U_{a_i})(\omega^U(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n))\right) (m') \\
&= {}_{\omega(\mathbf{a})}(\theta_{a_i,m'}) \left({}_{\omega(\mathbf{a})}(\eta U_{a_i})(\omega^U(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n))\right) \\
&= ({}_{\omega(\mathbf{a})}\phi)({}_{\omega(\mathbf{a})}\theta_{a_i,m}) \left({}_{\omega(\mathbf{a})}(\eta U_i)(\omega^U(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n))\right) \\
&= {}_{\omega(\mathbf{a})}\phi\left(\left(\omega^U(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)\right)_{a_i} (m)\right) \\
&= {}_{\omega(\mathbf{a})}\phi(\omega^{\mathcal{H}M}_{\mathbf{a},i}(m)).
\end{aligned}$$

Now let M be a left Z-module. If $t = t'$ is an identity of $\mathbf{V}$, then by lemma 9.2, we have $t^{\mathcal{H}M}_{\mathbf{a},i} = (t')^{\mathcal{H}M}_{\mathbf{a},i}$ for each i. Thus, $\mathcal{H}M$ is totally in $\mathbf{V}$. $\square$

The next theorem can be compared with the treatment of [11, corollary 8.8]:

Theorem 9.4. *The functors $\mathcal{G}$ and $\mathcal{H}$ are inverses.*

Proof. Let M be an A-module totally in $\mathbf{V}$. We will show that for each n-ary operation symbol ω, $\omega^{\mathcal{HG}M}_{\mathbf{a},i} = \omega^{M}_{\mathbf{a},i}$, for all $\mathbf{a}$ and i. Let $m \in {}_a(\mathcal{HG}M) = {}_a(\mathcal{G}M) = {}_aM$. We have

$$\begin{aligned}
\omega^{\mathcal{HG}M}_{\mathbf{a},i}(m) &= \left(\omega^U(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)\right)_{a_i} m \\
&= \left({}_{\omega(\mathbf{a})}(\eta U_{a_i})(\omega^{U_{a_i}}_{\mathbf{a}}(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n))\right) m \\
&= \left(\omega^{Z_{a_i}}_{\mathbf{a}}((a_1)_{a_i}, \dots, (a_{i-1})_{a_i}, (x)_{a_i}, (a_{i+1})_{a_i}, \dots, (a_n)_{a_i})\right) m \\
&= \theta^M_{a,m}(\omega^{Z_{a_i}}_{\mathbf{a}}((a_1)_{a_i}, \dots, (a_{i-1})_{a_i}, (x)_{a_i}, (a_{i+1})_{a_i}, \dots, (a_n)_{a_i})) \\
&= \omega^M_{\mathbf{a}}({}_{a_1}0_{a_i}m, \dots, {}_{a_{i-1}}0_{a_i}m, 1_{a_i}m, {}_{a_{i+1}}0_{a_i}m, \dots, {}_{a_n}0_{a_i}m) \\
&= \omega^M_{\mathbf{a}}({}_{a_1}0, \dots, {}_{a_{i-1}}0, m, {}_{a_{i+1}}0, \dots, {}_{a_n}0) \\
&= \omega^M_{\mathbf{a},i}(m).
\end{aligned}$$

Thus, $\mathcal{HG} = 1_{\mathbf{Ab}[A,\mathbf{V}]}$.

Now, let M be a left Z-module. We must show that for all $a, b \in A$, for all $z \in {}_aZ_b$, and for all $m \in {}_b(\mathcal{GH}M) = {}_bM$, we have the action of z on m the same in $\mathcal{GH}M$ as in M. It suffices to show this for z of the form $(u)_b$, since such elements generate ${}_aZ_b$ as a group, and the two actions are bilinear. Let $u = t^U(x, c_1, \dots, c_n)$, and denote $\langle b, c_1, \dots, c_n\rangle$ by $\langle b, \mathbf{c}\rangle$. In $\mathcal{GH}M$, we have

$$\begin{aligned}
(u)_b m &= \left(t^U(x, c_1, \dots, c_n)\right)_b m \\
&= \left({}_{u(b)}(\eta U_b)(t^{U_b}_{\langle b,\mathbf{c}\rangle}(x, c_1, \dots, c_n))\right) m \\
&= \left(t^{Z_b}_{\langle b,\mathbf{c}\rangle}(1^Z_b, {}_{c_1}0_b, \dots, {}_{c_n}0_b)\right) m \\
&= \theta^{\mathcal{H}M}_{b,m}\left(t^{Z_b}_{\langle b,\mathbf{c}\rangle}(1^Z_b, {}_{c_1}0_b, \dots, {}_{c_n}0_b)\right) \\
&= t^{\mathcal{H}M}_{\langle b,\mathbf{c}\rangle}(m, {}_{c_1}0, \dots, {}_{c_n}0) \\
&= t^{\mathcal{H}M}_{\langle b,\mathbf{c}\rangle,1}(m).
\end{aligned}$$

However, this is $\left(t^U(x, c_1, \dots, c_n)\right)_b m$ in M, by lemma 9.2. Thus, $\mathcal{GH} = 1_{Z\text{-}\mathbf{Mod}}$. $\square$

10. $\mathbb{Z}[A, \mathbf{V}]$ and Z_M

If M is an A-module totally in $\mathbf{V}$, then the action on Z on M yields a group homomorphism from ${}_aZ_b$ to ${}_a\operatorname{End}(M)_b = \mathbf{Ab}({}_bM, {}_aM)$ for each $a, b \in A$. These mappings comprise a ringoid homomorphism $f_M : Z \to \operatorname{End}(M)$, sending $(u)_b$ to $u^M_{\langle b\rangle}$.

Theorem 10.1. *We have*

1. *f_M is a ringoid homomorphism;*
2. *f_M has range in Z_M, i.e., $f_M(z) \in {}_b(Z_M)_a$ for all $z \in {}_bZ_a$; and*
3. *f_M, considered as a homomorphism from Z to Z_M, is* cofaithful *(i.e., one-one and onto on objects and onto on all hom-sets).*

Proof. (1): It suffices to show that given u, $v \in U$, we have $v^M_{\langle u(b)\rangle} u^M_{\langle b\rangle} = (vu)^M_{\langle b\rangle}$. But this follows from the fact that Φ^M_{Pol} is a clone homomorphism.

(2): $\mathrm{Pol}_1(A, \mathbf{V})$ is generated by the constant polynomials $a \in A$, and x. Thus we can prove that $f_M((u)_b) \in Z_M$ for all u, by showing that the generators satisfy this condition and that the subset of $\mathrm{Pol}_1(A, \mathbf{V})$ satisfying the condition is closed under the basic operations. We have $f_M((a)_b) = a^M_{\langle b\rangle} = {}_a0^{Z_M}_b$ for all a, and $f_M((x)_b) = x^M_{\langle b\rangle} = 1_{bM}$. Finally, given $\omega \in \Omega_n$, and $\mathbf{u} \in \mathrm{Pol}_1(A, \mathbf{V})^n$, we have $f_M(\omega^{\mathrm{Pol}_1(A,\mathbf{V})}(\mathbf{u})) \in {}_{\omega(\mathbf{u}(b))}(Z_M)_a$. For, $\omega^{\mathrm{Pol}_1(A,\mathbf{V})}(\mathbf{u}) = (\tilde{\Phi}(\omega))(\mathbf{u})$ by theorem 5.1(4), so that we have

$$\begin{aligned}
f_M((\omega^{\mathrm{Pol}_1(A,\mathbf{V})}(\mathbf{u}))_b)(m) &= f_M((\tilde{\Phi}(\omega)(\mathbf{u}))_b(m) \\
&= ((\tilde{\Phi}(\omega)(\mathbf{u}))^M_{\langle b\rangle}(m) \\
&= ((\tilde{\Phi}(\omega))^M \circ_{\mathbf{u}^A} \mathbf{u}^M)_{\langle b\rangle}(m) \\
&= (\tilde{\Phi}(\omega))^M_{\mathbf{u}(b)} \left((u_1)^M_{\langle b\rangle}(m), \dots , (u_n)^M_{\langle b\rangle}(m)\right) \\
&= \sum_i \left(\omega^M_{\mathbf{u}(b),i}\right) \left((u_i)^M_{\langle b\rangle}(m)\right).
\end{aligned}$$

(3): Obviously, f_M is one-one and onto on objects. Let $\omega \in \Omega_n$, and $\mathbf{a} \in A^n$, and consider $\omega^M_{\mathbf{a},i} \in {}_{\omega(\mathbf{a})}(Z_M)_{a_i}$. Let $\mathbf{w} = \langle a^A_1, \dots , a^A_{i-1}, x^A, a^A_{i-1}, \dots , a^A_n\rangle \in (\mathrm{Clo}_1 |A|)^n$. We have

$$\begin{aligned}
&f_M \left((\omega(a_1, \dots , a_{i-1}, x, a_{i+1}, \dots , a_n))_{a_i}\right)(m) \\
&\qquad = (\omega(a_1, \dots , a_{i-1}, x, a_{i+1}, \dots , a_n))^M_{\langle a_i\rangle} (m) \\
&\qquad =(\omega^M \circ_{\mathbf{w}} \langle a^M_1, \dots , a^M_{i-1}, x^M, a^M_{i+1}, \dots , a^M_n\rangle)_{\langle a_i\rangle}(m) \\
&\qquad =\omega^M_{\mathbf{a}}({}_{a_1}0, \dots , {}_{a_{i-1}}0, m, {}_{a_{i+1}}0, \dots , {}_{a_n}0) \\
&\qquad =\omega^M_{\mathbf{a},i}(m).
\end{aligned}$$

The range of f_M therefore contains all the generators of Z_M, and thus is onto on all hom-sets. ∎

11. J and R

For every a, $b \in A$, we have ${}_aZ_b = {}_a(\hat{\mathcal{M}}U_b/\mathcal{K}U_b)$. Define ${}_a\hat{Z}_b = {}_a(\hat{\mathcal{M}}U_b)$. It is easy to define a ringoid structure on $\hat{Z}$ by extending the composition of unary polynomials $\mathbb{Z}$-bilinearly. The natural maps from ${}_a\hat{Z}_b$ to ${}_aZ_b$ are abelian group homomorphisms and by theorem 8.1(6), constitute a ringoid homomorphism. Thus, the kernel groups ${}_aJ_b = {}_a(\mathcal{K}U_b)$ constitute a ringoid ideal J of $\hat{Z}$.

We want to give another description of the ideal J, by giving generators for each abelian group ${}_aJ_b$. For each $n \geq 0$, each n-ary polynomial Π, and each $b \in A$, we define

$$R_{\Pi,b} = \Pi^{U,\iota_A}(x, \dots, x) - \sum_{j=1}^{n} \Pi^{U,\iota_A}(b, \dots, b, x, b, \dots, b),$$

where in the sum over j, the j^{th} argument of Π^{U,ι_A} is x. For each a, $b \in A$, let ${}_aR_b$ be the subgroup of ${}_a(\hat{\mathcal{M}}U_b)$ generated by the elements $R_{\Pi,b}$, where $\Pi(b, \dots, b) = a$.

Theorem 11.1. *For all a, $b \in A$, ${}_aR_b = {}_aJ_b$.*

Proof. Let Π be an n-ary polynomial such that $\Pi(b, \dots, b) = a$. We will show that $R_{\Pi,b} \in {}_aJ_b$.

First, let $\mathbf{c}$ be an m-tuple of elements of A, and t an $(n + m)$-ary term, such that $\Pi = t^{\mathrm{Pol}_n A}(x_1, \dots, x_n, c_1, \dots, c_m)$. We have

$$t^U(x, \dots, x, c_1, \dots, c_m) = t^{U_b}_{\langle b,\dots,b,c_1,\dots,c_m\rangle}(x, \dots, x, c_1, \dots, c_m).$$

Also, by (7.1), we have for each $i \in \{1, \dots, n\}$,

$$t^{\hat{\mathcal{M}}U_b}_{\langle b,\dots,b,c_1,\dots,c_m\rangle,i}(x) - t^{U_b}_{\langle b,\dots,b,c_1,\dots,c_m\rangle}(b, \dots, b, x, b, \dots, b, c_1, \dots, c_m) \in {}_aJ_b,$$

where in the second term, x appears as the i^{th} argument. Finally, we have

$$t^{U_b}_{\langle b,\dots,b,c_1,\dots,c_m\rangle}(x, \dots, x, c_1, \dots, c_m) - \sum_{i=1}^{n} t^{\hat{\mathcal{M}}U_b}_{\langle b,\dots,b,c_1,\dots,c_m\rangle,i}(x) \in {}_aJ_b,$$

because the a-component of the natural map from $\hat{\mathcal{M}}U_b$ to $\mathcal{M}U_b$ sends this element to

$$t^{\mathcal{M}U_b}_{\langle b,\dots,b,c_1,\dots,c_m\rangle}(x/K, \dots, x/K, {}_{c_1}0, \dots, {}_{c_m}0) - \sum_{i=1}^{n} t^{\mathcal{M}U_b}_{\langle b,\dots,b,c_1,\dots,c_m\rangle,i}(x/K),$$

where K stands for ${}_b(\mathcal{K}U_b)$, and this is ${}_a0$.

Thus, we have

$$\begin{aligned}
R_{\Pi,b} &= \Pi^{U,\iota_A}(x, \dots, x) - \sum_{i=1}^{n} \Pi^{U,\iota_A}(b, \dots, b, x, b, \dots, b) \\
&= t^U(x, \dots, x, c_1, \dots, c_m) - \sum_{i=1}^{n} t^U(b, \dots, b, x, b, \dots, b, c_1, \dots, c_m) \\
&= t^{U_b}_{\langle b,\dots,b,c_1,\dots,c_m\rangle}(x, \dots, x, c_1, \dots, c_m) - \sum_{i=1}^{n} (t^{\hat{\mathcal{M}}U_b}_{\langle b,\dots,b,c_1,\dots,c_m\rangle,i}(x) + k_i) \\
&\in {}_aJ_b,
\end{aligned}$$

where $k_i \in {}_aJ_b$ for each i.

Since the abelian group generators of ${}_aR_b$ belong to the abelian group ${}_aJ_b$, we have ${}_aR_b \subseteq {}_aJ_b$.

To show ${}_aJ_b \subseteq {}_aR_b$, we will show that for each $b \in A$, the elements of ${}_a\mathcal{S}U_b$ (where $\mathcal{S}U_b$ is the A-set defined in section 7 that generates $\mathcal{K}U_b$) are contained in ${}_aR_b$, and also that for each b, the A-tuple of abelian groups ${}_aR_b$ is closed under action by elements of $Z_{\hat{\mathcal{M}}U_b}$, i.e., that the ${}_aR_b$ form an A-submodule of $\hat{Z}_b$.

We start with the fact that ${}_a*^{U_b} \in {}_aR_b$ for each $a \in A$. For, ${}_a*^{U_b} = a \in \operatorname{Pol}_0 A$. Next, let ω be an n-ary operation symbol, $\mathbf{a}$ an n-tuple of elements of A, and $\mathbf{u} \in {}_{\mathbf{a}}U_b$. We need to show that $\omega_{\mathbf{a}}^{U_b}(\mathbf{u}) - \sum_{i=1}^n \omega_{\mathbf{a},i}^{\hat{\mathcal{M}}U_b}(u_i)$ belongs to ${}_{\omega(\mathbf{a})}R_b$. Now, each u_i is an element of ${}_{a_i}U_b$, which is a subset of U, an algebra generated by x and elements of A. Thus, there are an m-tuple $\mathbf{d}$ of elements of A, and $(m+1)$-ary terms s_i, so that for each i we have

$$u_i = s_i^U(d_1, \dots, d_m, x).$$

For each i, define the n-ary polynomial

$$\Pi_i = s_i(d_1, \dots, d_m, x_i),$$

and let $\Pi' = \omega^{\operatorname{Pol}_n A}(\Pi_1, \dots, \Pi_n)$. Then we have

$$\omega_{\mathbf{a}}^{U_b}(\mathbf{u}) - \sum_{i=1}^n \omega_{\mathbf{a},j}^{\hat{\mathcal{M}}U_b}(u_i) = R_{\Pi',b} \in {}_{\omega(\mathbf{a})}R_b,$$

proving that $\mathcal{S}U_b \subseteq R_b$.

Now we must show that homomorphisms of the form $\omega_{\mathbf{c},i}^{\hat{\mathcal{M}}U_b}$, where ω is n-ary and $c_i = a$, send generators of ${}_aR_b$ to elements of ${}_{\omega(\mathbf{c})}R_b$. Consider the generator $R_{\Pi,b}$, where Π is m-ary and $\Pi(b, \dots, b) = a = c_i$. We have

$$\begin{aligned}\omega_{\mathbf{c},i}^{\hat{\mathcal{M}}U_b}(R_{\Pi,b}) &= \omega^U(c_1, \dots, c_{i-1}, \Pi^{U,\iota_A}(x, \dots, x), c_{i+1}, \dots, c_n) \\ &\qquad - \sum_{j=1}^m \omega^U(c_1, \dots, c_{i-1}, \Pi^{U,\iota_A}(b, \dots, b, x, b, \dots, b), c_{i+1}, \dots, c_n) \\ &= R_{\bar{\Pi},b},\end{aligned}$$

where in the summation, x appears in the j^{th} position, and $\bar{\Pi}$ is the m-ary polynomial

$$\omega^{\operatorname{Pol}_m(A,\mathbf{V})}(c_1, \dots, c_{i-1}, \Pi, c_{i+1}, \dots, c_n).$$

Thus, ${}_aJ_b \subseteq {}_aR_b$. □

12. Previous Constructions of Enveloping Ringoids

The second construction of the enveloping ringoid in [11], in the notation of this paper, defined $\mathbb{Z}[A, \mathbf{V}]$ as $\hat{Z}/R$, with only a slight difference in the description of the ideal R. Thus, the enveloping ringoid of this paper is the same as the one defined in [11].

13. Congruence-Modular Varieties

For congruence-modular $\mathbf{V}$, then results applying to that case [11, 12] yield some simplifications which shed light on the modulization functor and are helpful for finding the structure of the enveloping ringoid:

Theorem 13.1 ([11], Theorem C.10.4)**.** *Let P be a pointed A-overalgebra totally in $\mathbf{V}$, a congruence-modular variety. Let κ_P denote the kernel congruence of the projection homomorphism $\pi_P : A \ltimes P \to A$, and let κ denote $[\kappa_P, \kappa_P]$. Then $\operatorname{nat} \kappa_P$ factors as $\pi \circ \operatorname{nat} \kappa$ where $\pi : (A \ltimes P)/\kappa \to A$ is an onto homomorphism. Let $\iota = (\operatorname{nat} \kappa)\iota_P$. Then $\pi\iota = 1_A$. Let M denote $[\![(A \ltimes P)/\kappa, \pi, \iota]\!]$. Then $\langle M, \phi \rangle$ is a universal arrow from P to $\mathcal{U}$, where $\phi : P \to \mathcal{U}M$ is defined by ${}_a\phi : p \mapsto \langle a, p \rangle/\kappa$.*

Corollary 13.2. *Let P be a pointed A-overalgebra totally in $\mathbf{V}$, a congruence-modular variety. Then ηP is onto. I.e., for each $a \in A$, every element of ${}_a\mathcal{M}P$ is of the form ${}_a(\eta P)(p)$ for some $p \in {}_aP$.*

Thus, when $\mathbf{V}$ is congruence-modular, every element of ${}_a\mathbb{Z}[A, \mathbf{V}]_b$ is of the form $(u)_b$ for some $u \in \operatorname{Pol}_1(A, \mathbf{V})$.

Another consequence of this corollary is the fact that if P is a pointed A-overalgebra totally in $\mathbf{V}$ and $\mathbf{V}$ is congruence-modular, then $\mathcal{M}P$ is totally in $\mathbf{V}$. Of course, we proved this more generally for all $\mathbf{V}$ in §7, but it required more effort to prove without the assumption that $\mathbf{V}$ is congruence-modular.

Theorem 13.3. *Let A be an algebra in $\mathbf{V}$, a congruence-modular variety, and let d be a ternary difference term for $\mathbf{V}$. Let M be an A-module totally in $\mathbf{V}$. Then for each $a \in A$, and m, m', $m'' \in {}_aM$ we have*

$$m - m' + m'' = d^M(m, m', m'').$$

Corollary 13.4. *Let A be an algebra in $\mathbf{V}$, a congruence-modular variety, and let d be a ternary difference term for $\mathbf{V}$. If $b \in A$ and u, u', $u'' \in \operatorname{Pol}_1(A, \mathbf{V})$ are such that $u(b) = u'(b) = u''(b) = a$, then $(d^U(u, u', u''))_b = (u)_b - (u')_b + (u'')_b$.*

Proof. We have

$$\begin{aligned}
(d^U(u, u', u''))_b &= (d^{U_b}_{\langle a,a,a\rangle}(u, u', u''))_b \\
&= {}_a(\eta U_b)(d^{U_b}_{\langle a,a,a\rangle}(u, u', u'')) \\
&= d^{Z_b}_{\langle a,a,a\rangle}((u)_b, (u')_b, (u'')_b) \\
&= (u)_b - (u')_b + (u'')_b.
\end{aligned}$$

□

Underlying groups. In many cases of interest, a variety $\mathbf{V}$ has an underlying group functor. In such a case, given $A \in \mathbf{V}$ and a, $b \in A$, let $u = xa^{-1}b$ and $v = xb^{-1}a$. Then we have $uv = x$ and $vu = x$. Thus, $(u)_{v(b)}(v)_b = {}_a1^Z_b$, and $(v)_{u(a)}(u)_a = {}_b1^Z_a$. It follows that all the endomorphism rings ${}_aZ_a$ are isomorphic, and we have situations like those listed in table 1. See [12] for a more extensive discussion.

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